Each problem is worth 10 points, show all work and give adequate explanations for full credit. Please keep your work as legible as possible. Any similarity to actual persons is kinda funny.

1. Write the first four partial sums in the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

10

$s_1 = \frac{1}{2(1)-1} = \frac{1}{2-1} = 1$

$s_2 = 1 + \frac{1}{2(2)-1} = 1 + \frac{1}{4-1} = \boxed{1+\frac{1}{3}}$

$s_3 = 1 + \frac{1}{3} + \frac{1}{2(3)-1} = \boxed{1+\frac{1}{3}+\frac{1}{5}}$

$s_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{2(4)-1} = \boxed{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}}$

2. Determine whether the sequence $\left\{\frac{1}{3^n}+3\right\}$ converges or diverges, and if it converges find the limit.

10

converges sum converge on a number

$\lim_{n\to\infty} \frac{1}{3^n} + 3 = 3$

denominator gets infinetly big.

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ converges or diverges.

10

well, $\frac{1}{2n-1}$ looks to be greater than $\frac{1}{2n}$, so Comparison Test!

is $\frac{1}{2n-1} > \frac{1}{2n}$ for all n? (yes, denominator for $2n-1$ is smaller...)

So, if that's the case, we know $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$

is a p-series, with $p=1$. This series diverges

and, since $\frac{1}{2n-1} > \frac{1}{2n}$, by the Comparison test,

$\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges as well. Excellent

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{7^n}{n!}$ converges or diverges.

10

when we see factorials, we immediately think: Ratio Test!! (no, not the Rat Test, because rats are Evil, but rather the Ratio Test!!) :-)

$$\lim_{n\to\infty}\left|\frac{7^{n+1}}{(n+1)!}\cdot\frac{n!}{7^n}\right| =$$

$\frac{7^{n+1}}{7^n} = 7$

$\frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$\lim_{n\to\infty}\left|\frac{7}{n+1}\right| = \frac{7}{\infty} = 0$$

plug in ∞

and since $0 < 1$, we know that this series must Absolutely converge, and therefore converge because of our rules.

so

CONVERGES by the RATIO TEST!

Very Nice.

/

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges or diverges.

1) limit comparison test

Att.? ✓ $(-1)^n$

$\lim_{n\to\infty} = 0$? ✓ $\lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{\infty} = 0$

$f'(x) < 0$? ✓ $f(x) = (2n-1)^{-1}$ $f'(x) = -1(2)(2n-1)^{-2}$

$-$ $+$ $+$

$f'(x)$ is negative

The limit converges by test

Great

6. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges or diverges.

W

↳ integral test

u-sub

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3} \Rightarrow f(x) = \frac{1}{x(\ln x)^3} \Rightarrow \int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

$$= \int_2^{\infty} \frac{1}{u^3}\,du$$

$$= \frac{u^{-2}}{-2}\Bigg|_2^{\infty} \Rightarrow \frac{-1}{2(\ln x)^2}\Bigg|_2^{\infty} = 0 + \frac{+1}{2(\ln 2)^2}$$

$\approx \boxed{1.04}$

*$\boxed{\text{Converges}}$ by integral test

Well done.

7. Malcolm is a Calculus student at Rice University who's disappointed with his grade on his first exam. Malcolm says "Oh *dear*. I do seem to have scored rather less well than I would have liked. One question in particular remains irksome still. We were to determine the behavior of the series $\sum_{n=2}^{\infty} \frac{2-n}{\ln n}$, and I'm afraid I seem to have botched it terribly. I used the comparison test, since checking the first several terms made it quite clear that the terms in this series are less than those in $\Sigma\, 1/n^2$, and so all that should make it converge. I was really put out when the Prof returned the exam with no credit whatsoever on the problem and a note about using the Test for Divergence. I suppose I see his point about the Divergence and all, but I'm afraid I simply don't see what's wrong with *my* way."

W

Explain clearly to Malcolm, in terms he can understand, either what's wrong with his approach or how it can be reconciled with what his Professor said.

The terms a_n of the series are, indeed, less than those of Σ. However, Malcolm, you didn't take into account that the terms are becoming more and more negative, which means that the series diverges in the negative direction. Comparing this series with a completely positive series, such as $\Sigma \frac{1}{n^2}$, is invalid unless you use the absolute value of $\sum_{n=2} \frac{2-n}{\ln n}$, which is not less than $\Sigma \frac{1}{n^2}$.

Observe: $\left|\frac{2-n}{\ln n}\right| \overset{?}{\leq} \left|\frac{1}{n^2}\right| \qquad n \geq 2$

$$\frac{n-2}{\ln n} \overset{?}{\leq} \frac{1}{n^2}$$

Excellent

$$n^3 - 2n^2 \not\leq \ln n \rightarrow \text{test is invalid.}$$

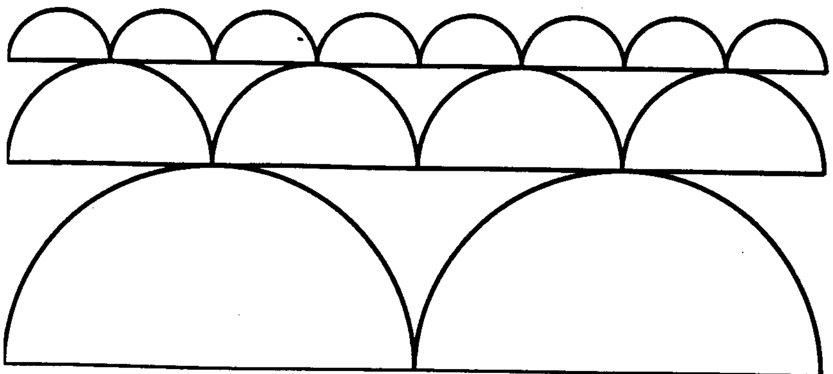
Now, the test for Divergence: $\lim_{n\to\infty} \frac{2-n}{\ln n} = \lim_{n\to\infty} \cancel{\frac{2}{\ln n}} - \frac{n}{\ln n}$

$$= \lim_{n\to\infty} -\frac{1}{(\frac{1}{n})} = \lim_{n\to\infty} -n \rightarrow \text{div}$$

$\boxed{\text{Series diverges}}$

8. [Finney/Thomas, 1990, p. 592] The figure below shows the first three rows of a sequence of rows of semicircles. There are 2^n semicircles in the n^{th} row, each of radius $1/2^n$. Find the sum of the areas of all the semicircles.

W



area semi circle $= \dfrac{\pi r^2}{2}$

$r = \dfrac{1}{2}^n$

· area = Area$_T$

$$A = \sum_{n=1}^{\infty} \frac{2^n \cdot \pi\left(\left(\frac{1}{2}\right)^n\right)^2}{2} = \sum_{n=1}^{\infty} \frac{\pi 2^n \left(\frac{1}{2}\right)^{2n}}{2} = \sum_{n=1}^{\infty} \frac{\pi 2^n \left(\frac{1}{4}\right)^n}{2}$$

Well done. geo seri

$a = \frac{\pi}{4}\left(\frac{1}{2}\right)^{n-1}$ geo form

$\Rightarrow A = \frac{1}{4}\pi + \frac{1}{8}\pi + \frac{1}{16}\pi$

$\boxed{\text{Area} = \frac{\pi}{2}}$

$\text{Sum} = \dfrac{\frac{\pi}{4}}{1-\frac{1}{2}} = \frac{\pi}{4} \cdot \frac{2}{1} = \frac{\pi}{2}$

9. Determine whether the series $\sum_{n=1}^{\infty} \sqrt[(n^2)]{2} - 1$ converges or diverges, where $\sqrt[(n^2)]{2}$ means taking the n^2 root of 2.

W

limit comparison test

$a_n = \sqrt[n^2]{2} - 1 \qquad b_n = \frac{1}{n^2}$

$$= \lim_{n\to\infty} \frac{\sqrt[n^2]{2} - 1}{\frac{1}{n^2}} \overset{\text{L.H.}}{\underset{\frac{0}{0}}{=}} \lim_{n\to\infty} \frac{2^{\frac{1}{n^2}} \ln 2 \left(\frac{-2}{n^3}\right)}{\frac{-2}{n^3}}$$

$$= \lim_{n\to\infty} 2^{\frac{1}{n^2}} \ln 2 = \ln 2 > 0 \qquad (= 2^0 = 1)$$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \ln 2 > 0$ so by the limit comparison test $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, that means $\sum_{n=1}^{\infty} \sqrt[n^2]{2} - 1$ converges also

Excellent

10. Prove that $\lim_{k\to\infty} \frac{a^k}{k!} = 0$. [Hint: You might want to start out by investigating $\Sigma\, a^k/k!$.]

RAT. TEST

$$\sum_{k=1}^{\infty} \frac{a^k}{k!} \rightarrow \lim_{k\to\infty} \left| \frac{a^{k+1}}{(k+1)!} \cdot \frac{k!}{a^k} \right| = \lim_{k\to\infty} \left| \frac{a}{k+1} \right| = 0 = L$$

by the RAT. Test, if $L < 0$, the series is CONVERGENT.

10

A theorem states that if $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

So, because the series $\sum \frac{a^k}{k!}$ is convergent, $\lim_{k\to\infty} \frac{a^k}{k!} = 0$.

Perfect

Extra Credit [this problem can replace your lowest-scoring problem on the exam]

a) Give an example of a series $\Sigma\, a_n$ which is divergent, but for which $\Sigma\, (a_n)^2$ is convergent.

$\Sigma \frac{1}{n}$ diverges (p-series $p \le 1$) $\quad \Sigma \left(\frac{1}{n}\right)^2 = \Sigma \frac{1}{n^2}$ converges (p-series $p > 1$)

b) Show that whenever $\Sigma\, a_n$ is convergent, $\Sigma\, (a_n)^2$ is convergent also.

Limit Comp. Test.

$\lim \frac{a_n}{b_n} = c$ & $c > 0$ both conv. or diverg.

10

$a_n = a_n^2$ $\quad a_n$

$b_n = a_n$

$$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n$$

if a_n converges, then $\lim_{n\to\infty} a_n$ converges & the limit comp. test says that a_n & a_n^2 do the same thing, so a_n^2 converges.

Well done