Calculus III Exam 2 Fall 2000 10/20/2000

Each problem is worth 10 points, show all work and give adequate explanations for full credit. Please keep your work as legible as possible.

1. Find the Taylor polynomial of degree 5 for the function f(x)=sin x centered at a=π.

10

n	$f^{(n)}$	$f^{(n)}(\pi)$
0	$\sin x$	0
1	$\cos x$	-1
2	$-\sin x$	0
3	$-\cos x$	$+1$
4	$\sin x$	0
5	$\cos x$	-1

$$0 + \frac{-1}{1!}(x-\pi) + \frac{0}{2!}(x-\pi)^2 + \frac{+1}{3!}(x-\pi)^3 + \frac{0}{4!}(x-\pi)^4 + \frac{-1}{5!}(x$$

$$= -\frac{x-\pi}{1!} + \frac{(x-\pi)^3}{3!} - \frac{(\pi-\pi)^5}{5!}$$

Great

2. Find the vertex, focus, and directrix of the parabola $8x^2 = -y$ and sketch its graph, indicating the positions of the focus and directrix on the graph.

10

$8x^2 = -y$

$x^2 = -\frac{1}{8}y$

$\frac{1}{4}4p = -\frac{1}{8}\cdot\frac{1}{4}$

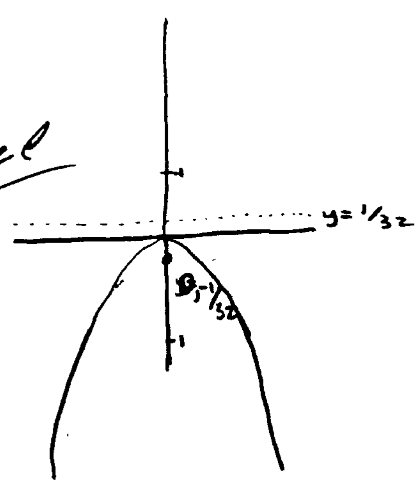
$p = -\frac{1}{32}$

focus $(0, \frac{-1}{32})$

directrix $y = \frac{1}{32}$

vertex $(0,0)$

Nice



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3. Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n\frac{(3x)^n}{2n}$.

The Rat Test! (Heeeeee...)

$a_n = (-1)^n \frac{(3x)^n}{2n}$ so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}(3x)^{\cancel{n+1}}}{\cancel{2}(n+1)}\cdot\frac{\cancel{2}n}{\cancel{(3x)^n}}\right|$

$= |3x| \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 3|x|$ as $n\to\infty$

so, $|x| < \frac{1}{3}$ means $R = \frac{1}{3}$

when $x=\frac{1}{3}$; $\sum_{n=0}^{\infty}(-1)^n\frac{(3(\frac{1}{3}))^n}{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n}$ converges by alternating series

$x=-\frac{1}{3}$; ... $\sum_{n=0}^{\infty}\frac{1}{2n}$ diverges by comparison

so $I = (-\frac{1}{3}, \frac{1}{3}]$ Great

4. A moose walks along a path given by the parametric equations $x=t^2+t$, $y=t^2-t$ (where t=0 of course represents the moment when the moose sees Officer Rebel). Set up an integral for the distance traveled by the moose (i.e., the arc length) between t=-1 and t=1.

10

$x = t^2 + t \qquad y = t^2 - t$

Formula for Arc Length

$L = \int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$$\boxed{L = \int_{-1}^{1}\sqrt{(2t+1)^2+(2t-1)^2}\,dt}$$

Moose on campus scares Dartmouth students

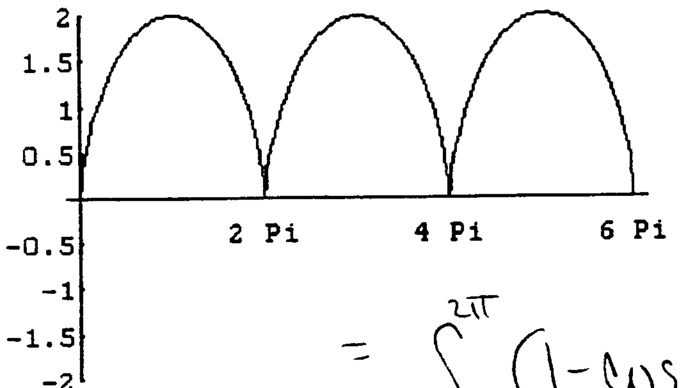
HANOVER, N.H. – Amid the normal flurry of activity on the Dartmouth College Green Thursday morning, students were surprised to fin a special visitor running through campus – a moose.

The moose ran through the West side of campus Thursday morning.

Although no one was hurt in the incident, there are some safety concerns when moose are running through populated areas, according to Safety and Security Officer Rebel Roberts, who responded to the call.

(taken from the Oklahoma Daily)

5. *Write* an integral for the area under *one arch* of the cycloid with parametric equations x = t - sin t, y = 1 - cos t. The plot below shows the curve for $0 \le t \le 6\pi$.



W

$$A = \int y(t) \cdot x'(t)\, dt$$

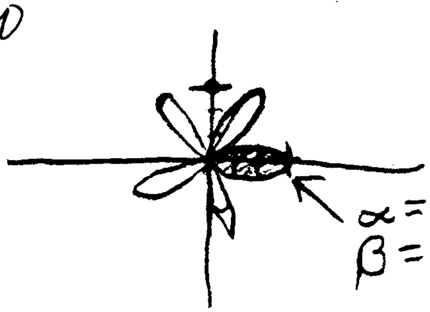
$$= \int_0^{2\pi} (1-\cos t)\left[(t - \sin t)'\right] dt$$

$$= \int_0^{2\pi} (1-\cos t)(1-\cos t)\, dt$$

Good

6. *Set up* an integral for the area inside one loop of the curve r = 3cos 5θ.

W



$r = 0$: $\cos 5\theta = 0$

$5\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$

$\theta = \pi/10, 3\pi/10, \ldots$

$\alpha = -\pi/10$

$\beta = \pi/10$

$$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$$A = \frac{1}{2}\int_{-\pi/10}^{\pi/10} (3\cos 5\theta)^2\, d\theta$$

$$\boxed{A = \frac{1}{2}\int_{-\pi/10}^{\pi/10} 9\cos^2(5\theta)\, d\theta}$$

Nicely done.

7. Eric is a calculus student at the University of Nebraska, and he's having some trouble with MacLaurin polynomials. Eric says, "Uh, wow. Those MacLaurin things are tough. My teacher keeps talkin' about how they tell us lots of things, but they don't really tell me much. On the last test we were supposed to use 'em to show that if you do sine of minus x it's the same as if you do the sine of x, but I don't have a clue how you'd do that."

10 Explain clearly to Eric how the MacLaurin series for sin (x) and sin (-x) relate to each other.

Since we already know the interval for the series sin x, we can easily find the series of sin (-x) by replacing all values for x, in the series with (-x).

Example:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(-x) = (-x) + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+}}{(2n+1)}$$

In essence this just changes the sign of the alternating series.

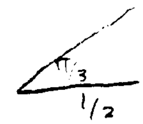
Excellent

Everything is ~~[illegible]~~ our friends :) !

8. Find the slope of the line tangent to $r = \cos\theta$ when $\theta = \pi/3$.

$$r = \cos\theta \qquad m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dr}{d\theta} = -\sin\theta$$



$r\cos\theta = \cos^2\theta$

$r\sin\theta = \cos\theta\sin\theta$

W

$$m = \frac{(-\sin\theta)(\sin\theta) + \cos^2\theta}{(-\sin\theta)(\cos\theta) - (\cos\theta\sin\theta)} = \frac{(-\sin^2(\pi/3)) + \cos^2(\pi/3)}{(-\sin\pi/3)(\cos\pi/3) - (\cos\pi/3\sin\pi/3)}$$

$$= \frac{-\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} \Rightarrow \frac{-\frac{3}{4} + \frac{1}{4}}{-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}} \Rightarrow \frac{-\frac{2}{4}}{-\frac{2\sqrt{3}}{4}} \cdot \frac{2}{\sqrt{3}} = \boxed{+\frac{1}{\sqrt{3}}}$$

Very well done.

9. Use the sixth degree MacLaurin polynomial for $f(x) = e^{x^2}$ to approximate the value of $\int_0^1 e^{x^2}\,dx$.

W

$$e^x = \sum_0^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \text{ etc.}$$

$$e^{x^2} = \sum_0^\infty \frac{(x^2)^n}{n!} = \sum \frac{x^{2n}}{n!}\ 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}$$

$$\int_0^1 e^{x^2}dx = \int_0^1 \left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}\right]dx = x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2} + \frac{x^7}{7\cdot 3!}\Big|_0^1$$

10 42

$$1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} - 0$$

$$= \boxed{1.45714 \approx 1.46}$$

Excellent

my calculator says it is 1.46265 hm.

Moral: The rest of the terms don't add up much!

10. Find the coordinates of the lowest point on the curve $x=t^3-3t$, $y=t^2+t+1$ [Hint: What should the slope of the tangent line be at the lowest point?].

10 at the lowest point, tangent line is horizontal $\Rightarrow \frac{dy}{dx}=0$

$$\frac{dy}{dx} = \frac{2t+1}{3t^2-3}$$

$$\frac{dy}{dx}=0 \Rightarrow 2t+1=0$$

$$\Rightarrow t=-\frac{1}{2}$$

$$x=\left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)$$

$$y=\left(-\frac{1}{2}\right)^2-\frac{1}{2}+1$$

Well done

$$\left(\frac{11}{8}, \frac{3}{4}\right)$$