Calculus III Exam 3 Fall 2000 12/1/2000

Each problem is worth 10 points, show all work and give adequate explanations for full credit. Please keep your work as legible as possible.

1. Write both vector and parametric equations for the line through (1,-4,3) and (3,0,2).

1st Find vector $(3,0,2) - (1,-4,3) = \langle 2,4,-1\rangle$ = vector

Equation for vector is $r = \langle point\rangle + t\,(vector)$

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$r = \langle 1,-4,3\rangle + t\,\langle 2,4,-1\rangle$

Great

Parametric equations are

$x = 1+2t,\quad y = -4+4t,\quad z = 3-t$

2. Find an equation for the plane containing the points (7,0,0), (0,3,0), and (0,0,10).

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$P = (7,0,0)$
$Q = (0,3,0)$
$R = (0,0,10)$

$\vec{PQ} = a = (0-7,\ 3-0,\ 0-0) = (-7,3,0)$

$\vec{PR} = b = (0-7,\ 0-0,\ 10-0) = (-7,0,10)$

$$a \times b = \begin{vmatrix} i & j & k \\ -7 & 3 & 0 \\ -7 & 0 & 10 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 0 \\ 0 & 10 \end{vmatrix} - \vec{j}\begin{vmatrix} -7 & 0 \\ -7 & 10 \end{vmatrix} + \vec{k}\begin{vmatrix} -7 & 3 \\ -7 & 0 \end{vmatrix}$$

$= i(30) - j(-70) + k(21)$

$= 30\vec{i} + 70\vec{j} + 21\vec{k}$

Nice.

$= 30(x-7) + 70(y-0) + 21(z-0) = 0$

$= 30x - 210 + 70y + 21z = 0$

$\Rightarrow 30x + 70y + 21z = 210$

3. Despite extremely tight security, an orange is thrown from the origin with an initial velocity of **8i+19j+5k** and subject to -9.8**k** acceleration. Give vector functions for the orange's velocity and position t seconds after being thrown.

$V_0 = 8\vec{i} + 19\vec{j} + 5\vec{k} \qquad a = 0\vec{i} + 0\vec{j} - 9.8\vec{k}$

We have $\frac{dv}{dt} = a \Rightarrow dv = a\,dt$

$\Rightarrow V = \int a\,dt$

$\Rightarrow V = -9.8t\vec{k} + 8\vec{i} + 19\vec{j} + 5\vec{k}$

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Great

$V = 8\vec{i} + 19\vec{j} + (-9.8t + 5)\vec{k}$

$\frac{dr}{dt} = v \Rightarrow r = \int v\,dt$

$\Rightarrow r = 8t\vec{i} + 19t\vec{j} + (-9.8t^2 + 5t)\vec{k}$

4. Alvin the Ant walks in a spiral path up around an ice-cream cone so that his path is given by the vector function **r**(t)=<tcos t, tsin t, 3t>. Show that the length of the path Alvin traverses from t=0 to t=2π is given by $\int_0^{2\pi} \sqrt{t^2+4}\,dt$ 10

$t\cos t = 1\cos t - t\sin t$

$t\sin t = \sin t + t\cos t$

$3t = 3$

$L = \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (3)^2}$

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$\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 9$

$1 = (\cos^2 t + \sin^2 t) + t^2\sin^2 t + t^2\cos^2 t + 9$

$1 + t^2(\sin^2 t + \cos^2 t) + 9$

$1 + t^2 + 9$

$= \int_0^{2\pi} \sqrt{t^2 + 10}$

Very Nice

5. Show that for any three-dimensional vectors **a** and **b**, **a** × **b** is perpendicular to **a**.

Therom: $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ (and to $\vec{b}$)

Proof: First do the cross product of $\vec{a} + \vec{b}$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2b_3 - a_3b_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)$$

Now dot the cross product result with $\vec{a}$.

$\vec{a} \cdot \langle \vec{a} \times \vec{b} \rangle$

$\langle a_1, a_2, a_3 \rangle \cdot \langle a_2b_3 - a_3b_2 - a_1b_3 + a_3b_1 + a_1b_2 - a_2b_1 \rangle$

$= a_2b_3a_1 - a_3b_2a_1 - a_1b_3a_2 + a_3b_1a_2 + a_1b_2a_3 - a_2b_1a_3$

$= 0$

When the dot product of two vectors equals zero, it can be stated the the two vectors are $\perp$

W

Very nicely done!

6. For what value of x is the vector <-1,5,x> perpendicular to the vector <2,4,-3>?

If they are $\perp$, then $\vec{a} \cdot \vec{b} = 0$

$\langle -1, 5, x \rangle \cdot \langle 2, 4, -3 \rangle = 0$

$(-1)(2) + (5)(4) + (x)(-3) = 0$

W

$-2 + 20 - 3x = 0$

$-3x = -18$

$x = 6$

Good

7. Jonathan is a Calc student at Kansas State University, and he's having some trouble with cross products. Jonathan says "Man, I just can't handle this stuff. We had this test question about, like, when you do the cross product of two unit vectors does the answer have to be a unit vector. Well, I knew that if you do like **i** crossed with **j** then you get **k**, and that's a unit vector, so I said it had to work. When we got the test back I got, like, practically no credit, and it's driving me nuts so I can't even focus on the game tomorrow!"

Explain to Jonathan why his reasoning is or isn't valid (so he can go into the game with a clear head and get beat fairly).

He doesnt deserve to get beat fairly going to KSU.

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well johnny boy first off unit vectors all have a magnitude of one, which means if you square the units, add them together and take the square root it best be one. So John say we had these unit vectors.

$\langle -1,0,0\rangle$ and $\langle 1,0,0\rangle$ and we cross them

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0,0)$$

Beautiful.

we'll get the vector $\langle 0,0,0\rangle$

and john what is the magnitude of that

well it is zero so does zero = one, Right

you are John it does not, so you dont always get a unit vector when you cross two other unit vectors.

$(y-1)(y-1) = y^2 - 2y + 1$

8. Reduce the quadric $x^2 + 8y = z^2 + 4y^2 + 8$ to standard form, classify the surface, and take a moment to visualize it mentally.

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$x^2 + 8y = z^2 + 4y^2 + 8$

$x^2 + 8y - z^2 - 4y^2 = 8 \Rightarrow x^2 - 4(y^2 - 2y + 1) - z^2 = 8 - 4$

$\frac{x^2}{4} - \frac{4(y-1)^2}{4} - \frac{z^2}{4} = \frac{4}{4}$

Standard form

$\frac{x^2}{4} - (y-1)^2 - \frac{z^2}{4} = 1$

Excellent

This is a Hyperboloid of two Sheets!

Two negatives

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9. The helix $\mathbf{r}_1(t)=\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ intersects the circle $\mathbf{r}_2(t)=\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\,\mathbf{k}$ at the point (1,0,0). Find the angle of intersection of these curves.

we can find the angle of intersect by finding the unit tangent vectors at that point and then using our equation $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ to find the equation.

1) Finding Unit Tangent Vectors

Helix: $\vec{r}_1(t) = \langle \cos t, \sin t, t\rangle$

$\vec{r}_1'(t) = \langle -\sin t, \cos t, 1\rangle$

but we need a t value. Since its pt (1,0,0) we know $z = 0 = t$ so $t = 0$

(also $\cos t = 1$, $\sin t = 0$)

$\vec{r}_1'(t) = \langle -\sin 0, \cos 0, 1\rangle$

$= \langle 0, 1, 1\rangle$

$|\vec{r}_1'(t)| = \sqrt{0+1+1} = \sqrt{2}$

$\therefore \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

2) Circle: $\vec{r}_2(t) = \langle \cos t, \sin t, 0\rangle$ at pt (1,0,0)

$y = \sin t = 0 \quad t = 0$

$\vec{r}_2'(t) = \langle -\sin t, \cos t, 0\rangle$

$= \langle -\sin 0, \cos 0, 0\rangle = \langle 0, 1, 0\rangle$

$|\vec{r}_2(t)| = \sqrt{0+1^2+0} = \sqrt{1} = 1$

a unit tangent vector already. cool!

3) Now we can plug our 2 vectors $\vec{a}\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$ + $\vec{b}\langle 0,1,0\rangle$ into our equation to find θ

$\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle \cdot \langle 0,1,0\rangle = (1)(1)\cos\theta$

unit tangent vectors.

$0(0) + (\frac{1}{\sqrt{2}})(1) + \frac{1}{\sqrt{2}}(0) = \cos\theta$

$\frac{1}{\sqrt{2}} = \cos\theta$

$\theta = \frac{\pi}{4} = $ or $45°$

Very well done!!

I guess they didn't have to be unit tangent vectors. But it just seemed safer that way... :)

10. Find the curvature of the helix **r**(t)=(a cos t) **i** + (a sin t) **j** + bt **k**.

$$k = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$r(t) = (a\cos t)i + (a\sin t)j + bt k$

$= \langle a\cos t, a\sin t, bt\rangle$

$r'(t) = \langle -a\sin t, a\cos t, b\rangle$

$r''(t) = \langle -a\cos t, -a\sin t, 0\rangle$

$$\frac{|\langle -a\sin t, a\cos t, b\rangle \times \langle -a\cos t, -a\sin t, 0\rangle|}{|\langle (a\sin t)^2, (a\cos t)^2, b^2\rangle|^3}$$

W

$$r'(t) \times r''(t) \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \vec{i}(0 + ba\sin t) - \vec{j}(0 + ba\cos t) + \vec{k}\,(a^2\sin^2 t + a^2\cos^2 t)$$

$$a^2(\underbrace{\sin^2 t + \cos^2 t}_{1})$$

$= |\langle ba\sin t, -ba\cos t, a^2\rangle| \Rightarrow$

$\sqrt{(ba\sin t)^2 + (-ba\cos t)^2 + (a^2)^2} \Rightarrow \sqrt{b^2a^2\sin^2 t + b^2a^2\cos^2 t + a^4} \Rightarrow$

$\sqrt{b^2a^2(\underbrace{\sin^2 t + \cos^2 t}_{1}) + a^4} \Rightarrow \sqrt{b^2a^2 + a^4} = \sqrt{a^2(b^2 + a^2)} = a\sqrt{b^2+a^2}$

$\Rightarrow |\langle -a\sin t, a\cos t, b\rangle| = \sqrt{(a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2\sin t^2 + a^2\cos^2 t + b^2}$

$\sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} \Rightarrow (\sqrt{a^2+b^2})^3 \Rightarrow (a^2+b^2)(\sqrt{a^2+b^2})$

put it all together: $\dfrac{a\cancel{\sqrt{b^2+a^2}}}{(a^2+b^2)(\cancel{\sqrt{a^2+b^2}})} \Rightarrow \boxed{\dfrac{a}{a^2+b^2}}$

Great Job