Exam 2 **Calculus 1** **10/11/2002**

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. Past returns do not assure future performance.

1. Compute the derivative of f(x) = x ln x - x.

$f(x) = x \cdot \ln x - x$

$f'(x) = 1 \cdot \ln x + x\left(\frac{1}{x}\right) - 1$

$= \ln x \cancel{+1} \cancel{-1}$

$= \ln x$

W

2. A table of values for f, g, f′, and g′ appears at right.
(a) If h(x) = f(g(x)), find h′(2).

x	f(x)	g(x)	f′(x)	g′(x)
1	3	2	4	6
2	1	3	5	1
3	7	2	8	0

$f'(g(2)) \cdot g'(2)$

$f'(3) \cdot 1$

$8 \cdot 1 = 8$

Good

W

(b) If $H(x) = \dfrac{f(x)}{g(x)}$, find H′(2).

$$\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

$$\frac{5 \cdot 3 - 1 \cdot 1}{(3)^2} = \frac{15 - 1}{9} = \frac{14}{9}$$

3. Write an **equation** for the line tangent to $f(x) = e^x$ when $x=1$.

$f(x) = e^x$ when $x=1$, $y=e$

$f'(x) = e^x$

$f(1) = e^1 = e$

$f'(1) = e^1 = e$

$$y - e = e(x-1)$$

[redacted]

Great

$$y = e(x-1) + e$$

$$y = ex - e + e$$

$$y = ex$$

4. For the implicitly defined ellipse $5x^2 - 6xy + 5y^2 = 16$,

a) Find y'.

$$\frac{d}{dx}\left(5x^2 - 6xy + 5y^2 = 16\right)$$

$$10x - 6(1 \cdot y + x \cdot 1 \cdot y') + 10yy' = 0$$

$$-6(y + xy') + 10yy' = -10x \qquad -6y - 6xy' + 10yy' = -10x$$

$$y'(-6x + 10y) = -10x + 6y \qquad y' = \left(\frac{-10x + 6y}{-6x + 10y}\right)\frac{2}{2} = \boxed{\frac{-5x + 3y}{-3x + 5y}}$$

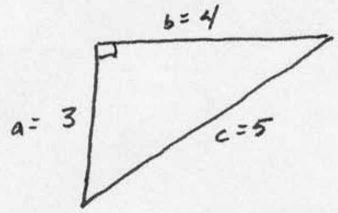
[redacted]

Excellent

b) Find the **slope** of the line tangent to the ellipse at the point (1,-1).

$$\frac{-5(1) + 3(-1)}{-3(1) + 5(-1)} = \frac{-5-3}{-3-5} = \frac{-8}{-8} = \boxed{1}$$

5. Jeb is driving north toward an intersection with a load of manure in the back of his truck at a rate of 60 miles per hour. Zeke is driving west toward the same intersection in a tanker truck full of toxic waste at a rate of 80 miles per hour. Obviously they're going to collide spectacularly, but putting that aside for the moment: At the instant when Jeb is 3 miles south of the intersection and Zeke is 4 miles east of the intersection, how fast are they approaching each other?



$\frac{da}{dt} = 60$

$\frac{db}{dt} = 80$

$a^2 + b^2 = c^2$

$2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}$

$2(3)(^{-}60) + 2(4)(^{-}80) = 2(5) \frac{dc}{dt}$

$-360 + {}^{-}640 = 10 \frac{dc}{dt}$

$\frac{dc}{dt} = {}^{-}100$ mi/hr

6. Show that the derivative of f(x) = csc x is f′(x) = -csc x cot x (Remember that csc x = 1/sin x).

W

$f(x) = \frac{1}{\sin x}$

$f'(x) = \frac{0 \cdot \sin x - \cos x \cdot 1}{(\sin x)^2}$

$= \frac{-\cos x}{(\sin x)^2}$

$= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$

$= -\cot x \csc x$

$= -\csc x \cot x$

Well done

7. Samantha is a calculus student at Enormous State University, and she's having some trouble with derivatives of trig functions. Samantha says "So, like, for that problem on the test about the derivative of cotangent, that was easy, right? I know cotangent is cos x over sin x, and since the derivative of cos x is -sin x and the derivative of sin x is cos x, then the derivative of cos x over sin x is -sin x over cos x. That means the derivative of cot x is -tan x."

Then Samantha's friend Chuck says "But here in my notes from class it says that the derivative of cot x is $-\csc^2 x$."

Critique Samantha's derivative. If it's right, explain why the class notes would say something different; if it's wrong, explain the mistake

Help Samantha by explaining, in terms she can understand, either how to convince her professor she's right, or how it is that terms less than 1 can have a limit equal to 1.

She is saying that the derivative of $\frac{\cos x}{\sin x}$ is $\frac{-\sin x}{\cos x}$.

That does seem right just by looking at it, But taking the derivative of quotients is not that easy. There is a special formula for quotients that says: $\left(\frac{F}{g}\right)' = \frac{F'g - g'F}{g^2}$. Therefore:

$$\left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)'\sin x - (\cos x)(\sin x)'}{(\sin(x))^2} = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

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So in conclusion, Samantha is wrong and should be slipped with a Ruler for saying "like" so much. :)

Yes

Proof that: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

8. Prove the quotient rule from the definition of the derivative (you may use the product rule as well if you find it convenient).

① Reciprocal Rule | Prove $\frac{1}{g} = \frac{-g'}{g^2}$ | Let $h(x) = \frac{1}{g(x)}$ ② $h(x) = 1/g(x)$

$$h'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1/g(x+h) - 1/g(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(x) - g(x+h)}{g(x+h)(g(x))} \cdot \left(\frac{1}{h}\right)$$ ← almost def of deriv

$$\cancel{h}(1/h)$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \cdot \lim_{h \to 0} \frac{g(x) - g(x+h)}{h}$$

$$= \frac{1}{(g(x))^2} \cdot -g'(x)$$

$$h'(x) = \frac{-g'(x)}{g^2}$$

$R(x) = h(x)f(x)$ } use product rule $fs' + f's$

$$R'(x) = h(x)f'(x) + h'(x)f(x)$$

$$R'(x) = \frac{1}{g(x)} f'(x) + \frac{-g'(x)}{(g(x))^2} \cdot f(x)$$

$$R'(x) = \frac{g(x)}{g(x)}\frac{f'(x)}{g(x)} + \frac{-g'(x)}{(g(x))^2} \cdot f(x)$$

$$R'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

W

Very nicely done!

9. Show that the derivative of y = sin^{-1}(x) is $y' = \frac{\;}{\sqrt{1-x^2}}$.

W

$$y = \sin^{-1}(x)$$

$$\sin y = \sin(\sin^{-1}(x))$$

$$\sin(y) = x$$

Implicit Diff.

$$(\cos y)\,y' = 1$$

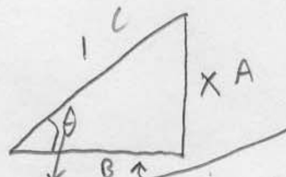
$$y' = \frac{1}{\cos y}$$

$$y' = \frac{1}{\cos(\sin^{-1}(x))}$$

$$y' = \frac{1}{\cos(\theta)}$$

$$y' = \frac{1}{\left(\frac{\sqrt{1-x^2}}{1}\right)}$$

$$\boxed{y' = \frac{1}{\sqrt{1-x^2}}}$$



$1^2 = x^2 + b^2$

$b^2 = 1^2 - x^2$

$b = \sqrt{1-x^2}$

Angle whose sin is $\frac{x}{1} = \frac{opp.}{hyp.}$

Excellent

10. Let f(x) = sin 2x. For what values of x does the line tangent to f have a slope of 1?

$f(x) = \sin(2x)$

$f'(x) = \cos(2x) \cdot 2 = 2\cos(2x)$

$f'(x) = 1$

$\Rightarrow \quad 2\cos 2x = 1$

$\Rightarrow \quad \cos 2x = \frac{1}{2}$

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$\Rightarrow \quad 2x = 2n\pi \pm \frac{\pi}{3}$ Excellent   ($\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ in 1^{st} quadrant)

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}$ or $-\frac{\pi}{6}, +\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots$

+5

Extra Credit (5 points possible):

Find a version of the product rule which tells how to do the derivative of a product of three functions, i.e. a formula for $[f(x)\cdot g(x)\cdot h(x)]'$

$A(x) = f(x)\cdot g(x)\cdot h(x)$

$A'(x) = [f(x)\cdot g(x)\cdot h(x)]'$

$= f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot [g(x)\cdot h(x)]'$

$= f'(x)\cdot g(x)\cdot h(x) + f(x)[g'(x)\cdot h(x) + g(x)\cdot h'(x)]$

$= f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot g'(x)\cdot h(x) + f(x)\cdot g(x)\cdot h'(x)$

Perfect