Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. All warrantees void in case of improper use.

1. Write the first four terms of the sequence \( \left\{ \frac{n-1}{n^2} \right\} \).
   \[
   a_1 = 0 \\
   a_2 = \frac{1}{4} \\
   a_3 = \frac{2}{9} \\
   a_4 = \frac{3}{16}
   \]

2. Write the first four partial sums of the series \( \sum_{n=1}^{\infty} \frac{n-1}{n^2} \).
   \[
   S_1 = 0 \\
   S_2 = \frac{1}{4} \\
   S_3 = \frac{1}{4} + \frac{2}{9} \\
   S_4 = \frac{1}{4} + \frac{2}{9} + \frac{3}{16}
   \]
   The partial sums just add each progressive \( a_n \).
   \[ s_n = a_n + a_{n-1} + a_{n-2} \ldots \]

3. Give an example of a series which is convergent but not absolutely convergent.
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\
   \text{Conditionally convergent} \\
   \text{Alt. Ser. Test} \\
   (a) \lim_{n \to \infty} \frac{1}{n} = 0 \\
   (b) \lim_{n \to \infty} \frac{n}{n} = 1
   \]
   Yes, converges by Out. Ser. Test
4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n}}$ converges or diverges.

Try comparison test.

\[
\frac{n^3 + n}{n^3 + n} > \frac{n^3}{n^3 + n} \\
\frac{\sqrt{n^3 + n}}{n} > \frac{\sqrt{n^3}}{n} \\
\frac{n}{n^3 + n} < \frac{1}{n^{3/2}}
\]

\[
\frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}
\]

**Excellent**

\[\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\] converges by the p-series test.

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges or diverges.

**Alt Series Test:**

\[\frac{1}{\sqrt{n}} > 0 \forall n\]

\[
\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}
\]

\[
0 \leq 1
\]

The series is dec.

\[\frac{1}{\sqrt{n}}\] fulfills all of the requirements for the alt series test, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.
6. Determine whether the series \( \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2} \) converges or diverges.

Try **Integral Test**

\[
\int_{1}^{\infty} \frac{e^{\frac{1}{x}}}{x^2} \, dx
\]

Let \( u = \frac{1}{x} \)

\( du = -\frac{1}{x^2} \, dx \)

\[
\lim_{b \to \infty} \int_{1}^{b} e^{\frac{1}{x}} \cdot \frac{1}{x^2} \, dx = -\left[ e^{\frac{1}{x}} \right]_{1}^{b} = -e^{-1} + e
\]

**Converges**

Since the integral converges, the given series converges as well by the integral test.

7. Write the third degree Taylor polynomial for the function \( f(x) = \ln x \) centered at \( a = 2 \).

\[
\begin{array}{c|c|c}
 n & f^{(n)}(a) & f^{(n)}(a) \times \frac{1}{n!} \\
0 & \ln 2 & \frac{\ln 2}{1!} \\
1 & \frac{1}{2} & \frac{1}{2!} \\
2 & -\frac{1}{2^2} & -\frac{1}{3!} \\
3 & \frac{1}{2^3} & \frac{1}{4!} \\
\end{array}
\]

\[
f(x) = \frac{f^{(0)}(a)}{0!} (x-a)^0 + \frac{f^{(1)}(a)}{1!} (x-a)^1 + \frac{f^{(2)}(a)}{2!} (x-a)^2 + \frac{f^{(3)}(a)}{3!} (x-a)^3
\]

\[
= \ln 2 + \frac{1}{2} (x-2) - \frac{1}{6} (x-2)^2 + \frac{1}{24} (x-2)^3
\]

\[
= \ln 2 + \frac{1}{2} (x-2) - \frac{1}{3} (x-2)^2 + \frac{1}{24} (x-2)^3
\]

\[
\boxed{\ln 2 + \frac{1}{2} (x-2) - \frac{1}{3} (x-2)^2 + \frac{1}{24} (x-2)^3}
\]

\[
-1 \cdot x^{-2} = -\frac{1}{x^2}
\]

\[
-1 \cdot x^{-3} = -\frac{1}{x^3}
\]

\[\frac{2}{3} \]

**Great**
8. Chaz is a calculus student at Enormous State University, and he’s having trouble with series. Chaz says “Ya know, I used to be pretty good at math, but this series crap is just outta control. What’s up with this thing where you do a bunch of work, and it turns out it’s no good? Like with that ratio test thing, you know? You do it, and you get 1, and they say that means you have to try something else. It’s like it’s just a conspiracy or something, because I did it for that one over n squared series, and it was like a total waste of time, and the series converges anyway. So why the heck don’t they just say if you get 1 from the ratio test, it’s gonna converge?”

Explain to Chaz, in terms he can understand, whether a 1 from the ratio test means that a series converges or not.

1. A 1 from the ratio test means absolutely nothing. It’s like asking a true/false question and getting “football.” In return, it tells you nothing either way, and it means you are asking the wrong answer-giver (in this case the ratio test). To demonstrate this, Chaz knows \( \frac{1}{n^2} \) converges and gets a 1 from the ratio test. However, we know \( \frac{1}{n} \) diverges, but it also gets a 1 from the ratio test:

\[
\lim_{n \to \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{n/1}{n/1 + 1/1} = 1,
\]

so we know that a 1 from the ratio test means nothing for convergence or divergence, because it would be telling you the same thing for a series we know is convergent and a series we know is divergent.

Very well done!
9. Find the radius of convergence of the power series \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n. \]

**RAT TEST**

\[ \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)^2 n^2} \right| = \lim_{n \to \infty} \frac{x^{n+1}}{x^n} \cdot \frac{1}{(n+1)^2} = \frac{1}{x} \cdot \frac{1}{(1+\frac{1}{n})^2} \]

because if the value of \( x \) that this drops out

The ratio test says for the series to converge,

\[ L = \frac{1}{|x|} < 1. \]

Therefore \( |x| < 1 \) \(-1 < x < 1\),

\[ L = \frac{1}{|x|} \]

Excellent

Radius of convergence \( R = 1 \)

10. Use the 6th degree polynomial for \( \sin(x^2) \) to approximate \( \int_0^1 \sin(x^2) \, dx \).

\[ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \]

\[ \sin (x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} = x^2 - \frac{x^6}{6} + \cdots \]

\[ \int_0^1 \sin (x^2) \, dx \approx \int_0^1 (x - \frac{x^6}{6}) \, dx = \left[ \frac{x^2}{2} - \frac{x^7}{42} \right]_0^1 = \frac{1}{2} - \frac{1}{42} = \frac{13}{42} \]

Extra Credit (This problem can replace your lowest other problem on the test): Find the sum of the series \( 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \ldots \), where the terms are of the form \( 1 \) over all possible natural number exponents on 2 or 3.

\[ \sum_{n=1}^{\infty} \frac{a_n + b_n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} \]

\[ \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{1}{2} \]

\[ \sum_{n=1}^{\infty} \frac{b_n}{2^n} = \frac{1}{3} \]

This is a geometric series

where \( r = \frac{1}{2} \) and \( a = \frac{1}{2} \) so

\[ S_{\infty} = \frac{1}{1 - \frac{1}{2}} = 1 \]

This also is a geometric series

where \( r = \frac{1}{2} \) and \( a = \frac{1}{3} \) so

\[ S_{\infty} = \frac{\frac{1}{3}}{1 - \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{1}{2} \]

\[ 2 \frac{1}{2} = \frac{5}{2} \]