Exam 1 Real Analysis 1 10/4/2002

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. Among the 112 faculty at a certain small liberal arts college, it is discovered that 57 are idiots, 78 are friendless, and 42 are mean people. Further examination reveals that 31 are both idiots and friendless, 13 are both idiots and mean, and 28 are both mean and friendless. If the only faculty member who is a friendless, mean, idiot is named Jon, then how many faculty members are neither friendless, mean, nor idiots?

2. Give an example of an odd function (you need not prove that it's odd, so long as it is).

3. State the definition of *convergence* of a sequence $\{a_n\}$.

4. State the definition of an *increasing* sequence.

5. Prove that if n is a natural number for which n^2 is odd, then n is also odd.

6. Prove that the sum of the first n odd natural numbers is n^2.

7. Prove from the definition that $\left\{\frac{1}{\sqrt{n}}\right\}$ converges to 0.

8. Prove or give a counterexample: If $\{a_n\}$ is a sequence which diverges to $+\infty$ and $\{b_n\}$ is another sequence, then $\{a_nb_n\}$ diverges to $+\infty$.

9. Using some or all of the axioms:

(A1) (Closure) $a+b$, $a \cdot b \in \mathbb{R}$ for any a, $b \in \mathbb{R}$. Also, if a, b, c, $d \in \mathbb{R}$ with $a=b$ and $c=d$, then $a+c=b+d$ and $a \cdot c=b \cdot d$.

(A2) (Commutative) $a+b = b+a$ and $a \cdot b = b \cdot a$ for any a, $b \in \mathbb{R}$.

(A3) (Associative) $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any a, b, c, $\in \mathbb{R}$.

(A4) (Additive identity) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a+0 = a$ for any $a \in \mathbb{R}$.

(A5) (Additive inverse) For each $a \in \mathbb{R}$, there exists an element -a in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (Multiplicative identity) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (Multiplicative inverse) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (Distributive) $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$ for any a, b, $c \in \mathbb{R}$.

(A9) (Trichotomy) For a, $b \in \mathbb{R}$, exactly one of the following is true: $a=b$, $a < b$, or $a > b$.

(A10) (Transitive) For a, $b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For a, b, $c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For a, b, $c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if a, $b \in \mathbb{R}$, then $a < b$ if and only if $-a > -b$. Be explicit about which axioms you use.

10. Prove that if $\{a_n\}$ converges to 0, then $\{(a_n)^2\}$ converges to 0.

Extra Credit (this problem can replace your lowest-scoring other problem): Prove that $\sqrt{2}$ is irrational.