Exam 1 **Real Analysis 1** 10/4/2002

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the limit L of a real-valued function f as x approaches ∞.

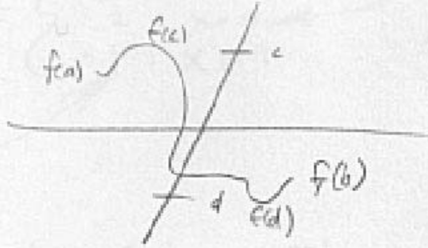
$\lim_{x \to \infty} |f(x) - L| < \varepsilon$

Let f(x) be a function with domain $D \subseteq R$ and D be not bounded above. We say that the limit of f(x) a x approaches ∞ is L iff for any $\varepsilon > 0$ the exists an M such that we have $|f(x) - L| < \varepsilon$ for all $x > M$ and $x \in D$.

Great

2. State the Extreme Value Theorem.

If f(x) is continuous on [a,b] then there exists $c, d \in [a,b]$ so that f(c) is the maximum value of f(x) on [a,b] and f(d) is the minimum value on f(x) on [a,b].



Exactly

3. State the Mean Value Theorem.

Suppose the following conditions are satisfied by a function f

a) f is continuous on $[a,b]$

b) f is differentiable on (a,b)

Then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(a)-f(b)}{a-b} = \frac{f(b)-f(a)}{b-a}$$

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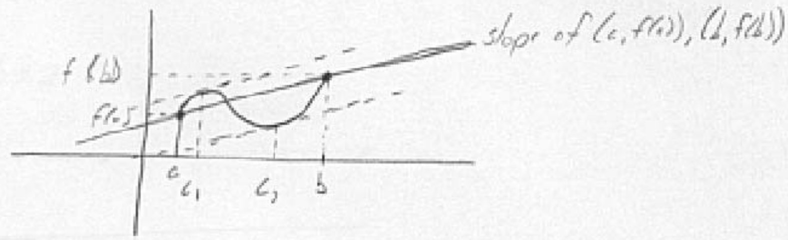
Nice!



4. Give an example of a function f:ℝ→ℝ which is discontinuous, but such that |f(x)| is continuous.

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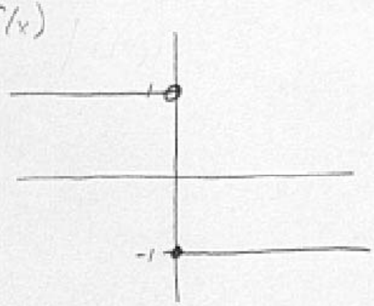
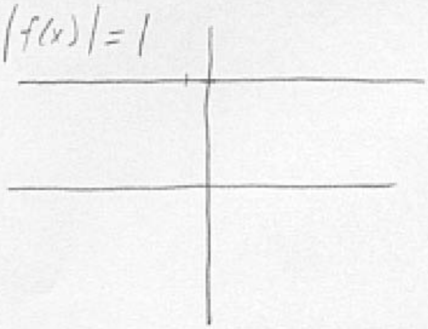
$$f(x) = \begin{cases} -1 & \text{when } x \geq 0 \\ 1 & \text{when } x < 0 \end{cases}$$

Great

f(x)



$|f(x)| = 1$



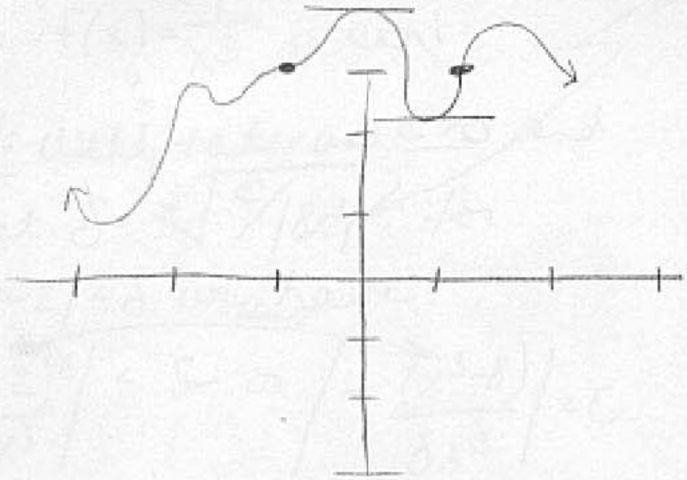
5. Prove or give a counterexample: If f:ℝ→ℝ is a differentiable function for which f′ is bounded, then f is bounded.

False: Let $f(x) = 2x$, f is differentiable, this is given, and $f'(x) = 2$, is bounded. But $f(x) = 2x$ is not bound over the entire set of reals. Thus false.

Great

then cont.

7. Prove or give a counterexample: If f:ℝ→ℝ is a differentiable function, f(-1) =3 and f(1)=3, there must be a point in the interval (-1,1) where f' is zero.



Diff + cont $f(a)=f(b)$

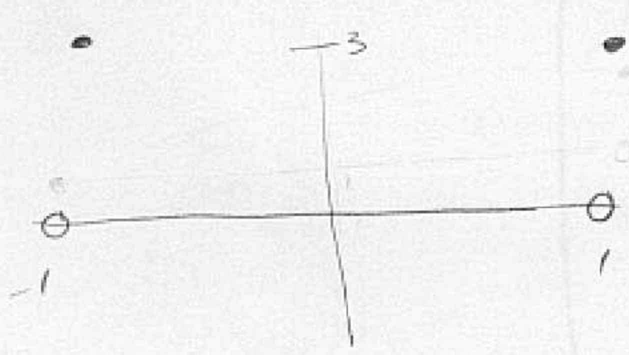
k between

$f'(c)=0$

True By Theorem because the theorm in the book says that if function f(x) is differentiable, then it is continuous. Yes!

And the Rolle's Theorem states that if f(x) is continuous + differentiable over interval $[a,b]$ and $f(a)=f(b)$ then there exists a $c \in (a,b)$ such that $f'(c)=0$.

Very nicely put!



8. Prove that $f(x) = 1/x^3$ is continuous at x=2.

From def.

Well given $\varepsilon > 0$ let $\delta = \min\{\frac{8}{7}\varepsilon, 1\}$

then if

$|x-2| < \frac{8}{7}\varepsilon$

then $\frac{7}{8}|x-2| < \varepsilon$ but we know for small ε, $\frac{x^2+2x+4}{8x^3} < \frac{7}{8}$

So $\frac{x^2+2x+4}{8x^3}|x-2| < \frac{7}{8}|x-2| < \varepsilon$

$\frac{x^2+2x+4}{8x^3}|x-2| < \varepsilon$

$\frac{1}{8x^3}|x^3-8| < \varepsilon$

$\left|\frac{x^3-8}{8x^3}\right| < \varepsilon$

$\left|\frac{1}{x^3} - \frac{1}{8}\right| < \varepsilon$

which is what we need to prove

f is continuous at x = 2. ∎

10 Excellent

Scratch work

$x^3 + 4x^2 - 4x - 2x^2 + 8$ $-4x$

$x-2(x^2 + 2x - 4)$

$\left|\frac{1}{x^3} - \frac{1}{8}\right| < \varepsilon$

$\left|\frac{8-x^3}{8x^3}\right| < \varepsilon$

$\frac{1}{8x^3}|x^3-8| < \varepsilon$

$\frac{x^2+2x+4}{8x^3}|x-2| < \varepsilon$

$\frac{7}{8}|x-2| < \varepsilon$

$|x-2| < \frac{8}{7}\varepsilon$

let $|x-2| < 1$

$x = 3$ or 1

$\frac{19}{216} < 1$

$x^3 + 2x^2 + 4x$

$-2x^2 - 4x - 8$

$$\begin{array}{r} x^2 + 2x + 4 \\ x-2\,\overline{)\,x^3 + 0x^2 + 0x - 8} \\ x^3 - 2x^2 \\ \hline +2x^2 \\ +2x^2 - 4x \\ \hline 4x \\ 4x - 8 \end{array}$$

6. Give an example of a function $f:(0,1)\to\mathbb{R}$ for which $f(x)$ is non-zero for infinitely many points in $(0,1)$, but $\lim_{x\to a} f(x) = 0$ for all $a \in (0,1)$.

$$\text{How about}\quad f(x) = \begin{cases} 0 & \text{for any irrational } x \\ \frac{1}{n} & \text{for any rational } x = \frac{m}{n} \end{cases}$$

It's non-zero for infinitely many points since there are infinitely many rationals in $(0,1)$, but the limit is zero as you approach any a, as explained in the book.

9. Prove that if $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then $\lim_{x\to a}(f-g)(x) = A-B$.

~~Let $\varepsilon>0$ be given.~~
~~Well, since $\lim_{x\to a} f(x)=A$, we know that for any $\varepsilon>0$, there's a δ_1 so that $0<|x-a|<\delta_1$ implies~~

Well, let $\varepsilon>0$ be given. Then since $\lim_{x\to a} f(x) = A$ and $\varepsilon/2>0$, we know there's a δ_1 so that $0<|x-a|<\delta_1$ implies $|f(x)-A|<\varepsilon/2$. Similarly since $\lim_{x\to a} g(x) = B$ there's a δ_2 so that $0<|x-a|<\delta_2$ implies $|g(x)-B|<\varepsilon/2$. Then let $\delta = \min\{\delta_1, \delta_2\}$. Now

$$\left|[f(x)-g(x)]-[A-B]\right| = \left|(f(x)-A)-(g(x)-B)\right| \le \left|f(x)-A\right| + \left|g(x)-B\right|$$

and since these last two terms can be made less than $\varepsilon/2$ each by taking $0<|x-a|<\delta$, we have that $(f-g)(x)$ is within ε of $A-B$ whenever x is within δ of a, as desired. □

10. Prove or give a counterexample: if f:ℝ→ℝ is an odd differentiable function, then f' is even.

odd $f(-x) = -f(x)$

even $f(-x) = f(x)$

Well, $f'(-x) = \lim_{h \to 0} \dfrac{f(-x+h) - f(-x)}{h}$

Since f is continuous then it doesn't matter which side we approach zero from, so we could come from the left side instead of the right. So if h is approaching from the left we have

$$f'(-x) = \lim_{h \to 0} \frac{f(-x-h) - f(-x)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-(x+h)) - f(-x)}{-h}$$

since $f(x)$ is odd

$$= \lim_{h \to 0} \frac{-f(x+h) + f(x)}{-h}$$

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$$= \lim_{h \to 0} \frac{-[f(x+h) - f(x)]}{-h}$$

Very nicely done!

the negatives cancel and

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(-x) = f'(x)$

so the derivative of an odd function is even ∎