Problem Set 9 Real Analysis 1 Due 12/6/2002

Each problem is worth 5 points. Adequate demonstration is required for full credit.

1. Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ for which an upper sum on [0,1] with four subdivisions gives three times the value of $\int_0^1 f\,dx$.

Example: Well, there are many good candidates, most of them "spiky" shapes of one sort or another, but perhaps the simplest is to let f(x) = 9/4 when x=1/8, and 1 everywhere else. □

2. Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ for which a lower sum on [0,1] with four subdivisions is zero even though the value of $\int_0^1 f\,dx$ is not zero.

Example: Well, again there lots of ways to do this, but consider f(x) = sin(8π x) + 1 for one clean example.

3. Prove or give a counterexample: If functions f and g are not Riemann integrable, then f+g is not Riemann integrable. □

Counterexample: Well, let f(x) = 1 at any rational and 0 at any irrational, and g(x) = 0 at any rational and 1 at any irrational. Then f+g(x) = 1 everywhere, which is Riemann integrable although f and g are not. □

4. Prove or give a counterexample: If $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function, then f is Riemann integrable on any interval [a,b].

Proof: Well, if f is differentiable then it's continuous by a theorem from a while back. And if f is continuous then it's Riemann integrable by a theorem in chapter 6, so any differentiable function has to be integrable. □

5. Prove or give a counterexample: If $f:\mathbb{R}\to\mathbb{R}$ is a Riemann integrable function on any interval [a,b], then f is differentiable.

Counterexample: Well, how about the example I used in problem 1. □

6. Prove that a constant function f(x) = c, for $c \in \mathbb{R}$, is Riemann integrable on any interval [a,b].

Proof: Well, take any partition of [a,b]. The upper sum on this partition will be (b-a)·c, and the lower sum will also be (b-a)·c, since in both cases on any subinterval the highest and lowest heights the function take on will be c. Then the upper and lower sums are always equal, so the function is Riemann integrable. □