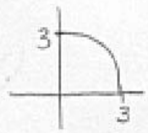
Exam 4a Calculus 3 11/25/2003

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}(x,y) = \langle 6xy, 3x^2+3y^2\rangle$ and C is the first-quadrant portion of a circle (centered at the origin) from (3,0) to (0,3).

$3x^2y + y^3$

a b



For $F(x,y) = \langle 6xy, 3x^2+3y^2\rangle$, $f(x,y) = 3x^2y + y^3$ is a potential function; therefore, I can use the Fund theorem.

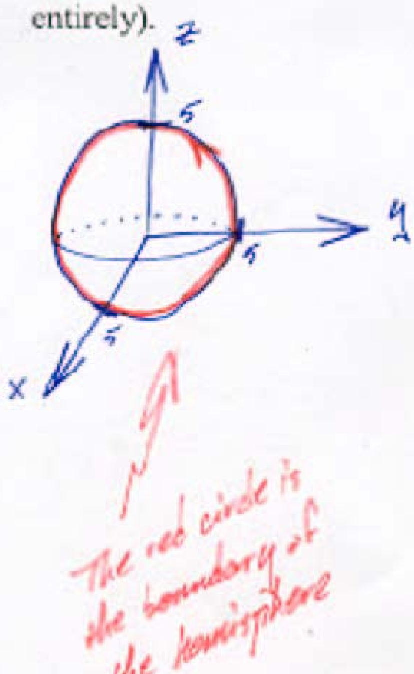
$= f(b) - f(a)$

$= f(0,3) - f(3,0)$

$= 27 - 0 = \boxed{27}$

Wonderful!

2. If $\mathbf{F}(x,y,x) = \langle 3, 5xy, z^2\rangle$ and S is the half of a sphere of radius 5 centered at the origin for which $x \geq 0$, **say what line integral you could work out instead** which is equivalent to the surface integral $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$, and parametrize the path involved (but you don't need to work it out entirely).



We'd work out $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}$ is the original $\langle 3, 5xy, z^2\rangle$ vector field and C is the circle $y^2+z^2=5^2$ in the plane $x=0$, parametrized by:

$x(t) = 0$
$y(t) = 5\cos t$
$z(t) = 5\sin t$

or

$\vec{r}(t) = \langle 0, 5\cos t, 5\sin t\rangle$

for $0 \leq t \leq 2\pi$

3. Compute $\nabla\times\mathbf{F}$ for $\mathbf{F}(x,y,z) = <3x,0,3y>$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

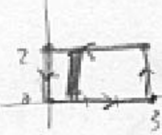
Here $\langle P, Q, R\rangle = \langle \overset{P}{3x}, \overset{Q}{0}, \overset{R}{3y} \rangle$

$$\nabla \times F = \langle 3-0, 0-0, 0-0 \rangle$$

$$= \underline{\langle 3, 0, 0 \rangle}$$

Great

4. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 3xy\mathbf{i} + 6x^2\mathbf{j}$ and C is the path consisting of four line segments joining the points (0,0), (3,0), (3,2), and (0,2) in that order.



$$\vec{F}(x,y) = \langle \overset{P}{3xy}, \overset{Q}{6x^2} \rangle$$

Greene's Theorem:

$$\int_C \vec{F}\cdot d\vec{r} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint (12x - 3x)\, dA$$

$$= \underline{\int_0^3\int_0^2 9x\, dy\,dx}$$

Excellent!

$$= \int_0^3 9xy\Big|_0^2 dx = \int_0^3 18x\ dx$$

$$= 9x^2\Big|_0^3$$

$$= \boxed{81}$$

5. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ and S is the portion of $z = x^2 + y^2$ between the planes $z = 1$ and $z = 9$.

Limits are circles with radius 1 and 3

① $x(u,v) = u \qquad y = (u,v) = v \qquad z = u^2 + v^2$

$\bar{r}(u,v) = \langle u, v, u^2+v^2 \rangle$

② $\bar{r}_u = \langle 1, 0, 2u \rangle \qquad \bar{r}_v = \langle 0, 1, 2v \rangle.$

$$\bar{r}_u \times \bar{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u)\hat{i} - j(2v) + k(1)$$

$$= \langle -2u, -2v, 1 \rangle$$

③ $F(\bar{r}(u,v)) = \langle 2u, 2v, 1 \rangle$

10

Well done

④ $$\iint_D F(\bar{r}(u,v)) \cdot (\bar{r}_u \times \bar{r}_v)\, dA$$

$$= \iint_D \langle 2u, 2v, 1 \rangle \cdot \langle -2u, -2v, 1 \rangle \, dA$$

$$= \iint_D -4u^2 - 4v^2 + 1 \quad dA$$

$$= \int_0^{2\pi}\int_1^3 (1 - 4r^2)\, r\,dr\,d\theta = \int_0^{2\pi}\int_1^3 (r - 4r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} - r^4 \Big|_1^3 \right) d\theta = 2\pi \cdot \left[\left(\frac{3^2}{2} - 3^4 \right) - \left(\frac{1}{2} - 1 \right) \right] = 2\pi \cdot 76\pi$$

$$= -152\pi$$

6. Prove that if f(x,y,z) is a function with continuous second-order partial derivatives, then curl(∇f) = **0**. Make it clear how the requirement that the partials be continuous is important.

$f(x,y,z)$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\text{curl } \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{curl}(\nabla f) = \left\langle \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right\rangle$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

these two are the same if have continuous 2nd order partial derivatives, so they cancel each other out & we get:

$$\text{curl}(\nabla f) = \langle 0, 0, 0 \rangle = \vec{0}.$$

Excellent!

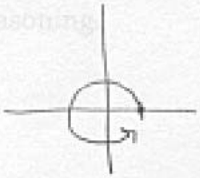
7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I am so totally confused by this class. I mean, I can work out a lot of the problems, but I totally don't understand what any of it means. I guess it mostly doesn't really matter, since our exams are all multiple choice, but there was this one on the old exam I got from the files in my boyfriend's frat where, like, it was a surface integral, and since they set it up by parametrizing the surface a totally different way than I did, their setup looked different than mine. It was multiple choice, and you were only supposed to set it up, so since my setup was different from theirs I totally didn't know which answer to mark. How can you tell?"

Explain as clearly as possible to Bunny what relationships might hold between surface integrals set up using different parametrizations of the same portion of the same surface.

"Regardless of the notation you use, or the parametrization you do, providing it is the same portion of the same surface, the final answer will be the same. It's like changing coordinate systems from Cartesian to spherical in order to make volume of a sphere calculations easier – the answer's the same, the surface is the same, the problem's the same. Different parametrizations only allow one (maybe) to compute the answer more easily."

Excellent.

8. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where F(x,y) = <-x, y> and C is a circular path (centered at the origin) beginning at (1,0) and traversing *n* quarter-circles (where, for instance, traversing 8 quarter-circles means passing twice around a circle).



$\vec{F}(x,y) = \langle -x, y\rangle$

$x(t) = \cos t$
$y(t) = \sin t \qquad 0 \le t \le n\pi/2$

$\vec{r}(t) = \langle \cos t, \sin t\rangle \qquad \vec{F}(\vec{r}(t)) = \langle -\cos t, \sin t\rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$

$\int_0^{n\pi/2} \langle -\cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt$

$\int_0^{n\pi/2} (\cos t \sin t + \sin t \cos t)\, dt$

$\int_0^{n\pi/2} (2\cos t \sin t)\, dt$

$2\int_0^{n\pi/2} \cos t \sin t \, dt \qquad \sin t = u \qquad du = \cos t\, dt$

$2\int_0^{n\pi/2} u\, du$

$2\left(\frac{u^2}{2}\right]_0^{n\pi/2 \text{ at}}$

$2\left(\frac{\sin^2 t}{2}\right]_0^{n\pi/2}$

$2\left(\frac{\sin^2(n\pi/2)}{2} - \frac{\sin^2 0}{2}\right)$ Nice!

$\sin^2(n\pi/2)$

9. Suppose that **F** is a vector field whose divergence is zero everywhere and S is a tetrahedron consisting of the four sides S_1, S_2, S_3, and S_4. If you know that $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = 2$, $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = 2$, and $\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = 2$, then what can be said about $\iint_{S_4} \mathbf{F}\cdot d\mathbf{S}$? Explain your reasoning.

We are working with an enclosed surface (the tetrahedron)

so we know

$$\iint_{S_t} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\cdot dv$$

total surface

which, by the information given means

$$\iint_{S_t} \vec{F}\cdot d\vec{S} = \iiint_E 0 \; dv = 0$$

the total surface integral of the tetrahedron is equal to zero, and can be given from the sum of it's sides.

that is,

$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} + \iint_{S_4} \vec{F}\cdot d\vec{S} = \iint_{S_t} \vec{F}\cdot d\vec{S}$$

$$2 + 2 + 2 + \iint_{4} \vec{F}\cdot d\vec{S} = 0$$

By the simplest of algebra,

Beautiful.

$$\iint_{S_4} \vec{F}\cdot d\vec{S} = -6$$

10. We talked in class about checking to see if a vector field $F(x,y) = <P(x,y),Q(x,y)>$ is conservative by comparing $P_y(x,y)$ and $Q_x(x,y)$. The analogous question for three-dimensional vector fields is a bit more involved: How could you test whether a vector field $F(x,y,z) = P(x,y,z),Q(x,y,z),R(x,y,z)>$ is conservative (apart from just seeking a potential function)?

*you could do the curl of the vector field, and if it equals zero, then it is conservative.

From my proof in #6 we know that $\text{curl}(\nabla f) = 0$, right? Well we also know that if $\vec{F}$ is conservative the $\vec{F} = \nabla f$. So therefore just replace $\vec{F}$ into $\text{curl}(\nabla f) = 0$ to get $\text{curl}(\vec{F}) = 0$.

Well put.