Exam 1 Calc 2 9/10/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int \sin t \, dt$.

$$\int \sin t \; dt = \boxed{-\cos t + C}$$

Excellent

$$\left[-\cos t + C\right]' = \sin t$$

2. Find the area below the curve f(*x*) = $3/x^2$ but above the *x* axis between *x* = 1 and *x* = 2.

$$\int_1^2 \frac{3}{x^2} = 3\int_1^2 (x^{-2}) = -3x^{-1}\Big|_1^2 \; -3(2)^{-1} - (-3\cdot(1)^{-1}) = -\frac{3}{2} - (-3\cdot 1) = -\frac{3}{2} + 3 = \boxed{\frac{3}{2}}$$

Justification

$$\left[-3x^{-1}\right]' = 3x^{-2} \text{ or } \frac{3}{x^2}$$

Great

3. Evaluate $\int \frac{y}{y^2+4}\,dy$.

let $u = y^2+4$

$\frac{du}{dy} = 2y$

$du = 2y\,dy$

$\frac{du}{2y} = dy$

$= \int \frac{\cancel{y}}{u}\,\frac{du}{2\cancel{y}}$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + C$

$= \boxed{\frac{1}{2}\ln|y^2+4| + C}$

Good

4. Evaluate $\int \frac{1}{t^2+4t+5}\,dt$.

$\int \frac{1}{(t^2+4t+4)+5-4}\,dt$

$\int \frac{1}{(t+2)^2+1}\,dt$ let $u = t+2$, $du = dt$, let $a = 1$

$\int \frac{1}{u^2+a^2}\,du$

$\frac{1}{a}\arctan\frac{u}{a} + C$

$= \boxed{\arctan(t+2) + C}$

Well done!

5. If $F(x) = \int_1^x \frac{1}{\sqrt{1+t^3}}\,dt$,

a) What is F(1) and why?

b) What is F′(2) and why?

(a) $\int_1^1 \frac{1}{\sqrt{1+t^3}}\,dt$ = no area because the interval is going nowhere. It is going from 1 to 1 and that distance is zero.

(b) $F'(2) = \int_1^2 \frac{1}{\sqrt{1+t^3}}\,dt$

Great!

$F'(2) = \frac{1}{\sqrt{1+t^3}} = \frac{1}{\sqrt{1+2^3}} = \frac{1}{\sqrt{9}} = \boxed{\frac{1}{3}}$

The derivative of the antiderivative of $\frac{1}{\sqrt{1+t^3}}$ is $\frac{1}{\sqrt{1+t^3}}$ and to find $F'(2)$ you substitute 2 in for the t and the answer comes out as $\frac{1}{3}$.

6. Use the trig substitution $x = 2\sin\theta$ to show that the integral $\int x^2\sqrt{4-x^2}\,dx$ is equivalent to the integral $16\int \sin^2\theta\cos^2\theta\,d\theta$.

$\int x^2\sqrt{4-x^2}\,dx$ let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\int (2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}\,2\cos\theta\,d\theta$

$\cos^2\theta = 1-\sin^2\theta$

$\int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta$

$8\int(\sin^2\theta)2\sqrt{(1-\sin^2\theta)}\cos\theta\,d\theta$

$16\int\sin^2\theta\sqrt{\cos^2\theta}\cos\theta\,d\theta$

$16\int\sin^2\theta\cos\theta\cos\theta\,d\theta$

$\boxed{16\int\sin^2\theta\cos^2\theta\,d\theta}$

Great

7. To celebrate Halloween, Jon plans to launch pumpkins across the quad from his office window (which is 16 feet above the ground). He wants the pumpkins to reach their maximum height after 2 seconds of flight.

a) What initial vertical velocity should the pumpkin be launched with to achieve this?

b) What will the pumpkin's velocity be when it hits the ground?

a) $a(t) = -32$

$v(t) = -32t + v_0$

$h(t) = -16t^2 + v_0 t + 16$

Then for the maximum height to occur when $t=2$,

$0 = -32(2) + v_0$

$v_0 = 64$

So the launch velocity should be 64 ft/sec^2.

b) It hits the ground when $h(t)=0$, so when

$0 = -16t^2 + 64t + 16$

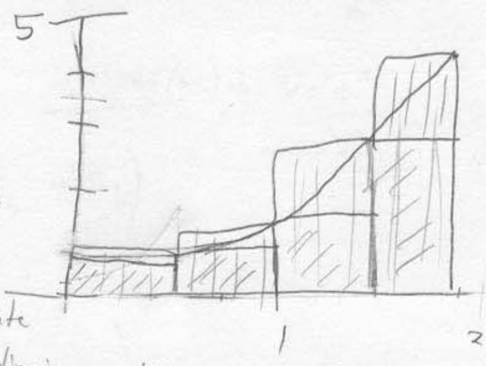
$t = \sqrt{5} + 2 \approx 4.236$ seconds

$v(\sqrt{5}+2) \approx -71.55$ ft/sec

8. Biff is a calculus student at a large state university, and he's having some trouble. Biff says "Man, calculus sucks. This integral stuff just makes no sense. I mean, there's left-handed stuff and right-handed stuff and middle-handed stuff and all that, and it's just crazy. There was this question on our quiz, like they gave us this function with the square root of this one plus x to the fourth stuff, and we had to integrate it from zero to two, and it asked if these different estimates was too high or too low or whatever. But I don't have one of those fancy calculators that integrates, so there's like no way I can know if those estimates are too high or too low, right? Am I just supposed to guess? What kinda crap is that?"

Explain clearly to Biff how he could know whether to expect the left, right, and midpoint approximations to be high or low without needing to know the true value of the integral.

$\int_0^2 \sqrt{1+x^4}\,dx$

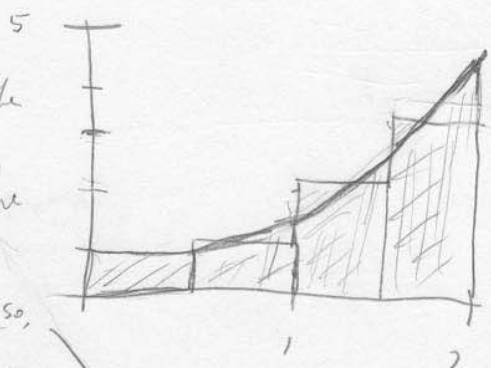


this graph is ascending and is concave up.

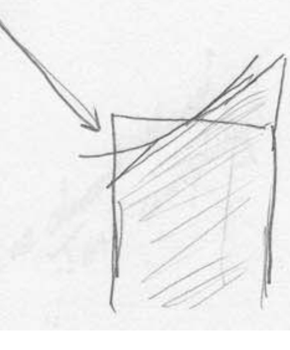
if you took a LEFT(4) approximation, the estimate will be lower than the exact value, because the function is <u>ascending</u>, leaving room under it for error.

x	y
1	$\sqrt{2}$
2	4.123
3	9.055

if you took a RIGHT(4) approximation, the estimate will be <u>higher</u>, because the ascending function makes the RIGHT estimate stick <u>out</u> to the left.



if you took a middle approximation, you can expect the estimate to be <u>lower</u> than the exact value, because the graph is conc<u>ave up</u>, and taking the midpoint is the same as making a <u>tangent line</u> trapezoid li<u>ke</u> so, and the curve sweeps away from the tangent line, leaving space for error; that is, area which is unaccounted for by the trapezoid.



Exactly!

9. Show how the integration formula

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}e^{ax}\left[a\sin(bx) - b\cos(bx)\right] + C$$

can be obtained using the techniques of integration we know.

$$\int e^{ax}\sin(bx)dx = e^{ax}\cdot\frac{-1}{b}\cos bx - \int a e^{ax}\cdot\frac{-1}{b}\cos bx\,dx$$

let $u = e^{ax}$ $\quad v = \frac{-1}{b}\cos bx$

$u' = a\cdot e^{ax}$ $\quad v' = \sin bx$

$$= \frac{-1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

$u = e^{ax}$ $\quad v = \frac{1}{b}\sin bx$

$u' = a\cdot e^{ax}$ $\quad v' = \cos bx$

$$= \frac{-1}{b}e^{ax}\cos bx + \frac{a}{b}\left[e^{ax}\cdot\frac{1}{b}\sin bx - \int a e^{ax}\cdot\frac{1}{b}\sin bx\,dx\right.$$

$$= \frac{-1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

$$\int e^{ax}\sin bx\,dx = \frac{-1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

Look! The same integral showed up again! Let's move it to the other side and solve!

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx = \frac{-1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx + C$$

$$\left(\frac{b^2+a^2}{b^2}\right)\cdot I = \frac{-1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx + C$$

$$I = \frac{-b}{a^2+b^2}e^{ax}\cos bx + \frac{a}{a^2+b^2}e^{ax}\sin bx + C$$

Where "I" represents the integral we were working out.

10. Our table of integrals doesn't have a formula for $\int \frac{1}{(x-a)(x-b)(x-c)}dx$. Show how one can be found using the techniques of integration we know.

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \quad \text{or} \quad 1 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$$

If $x=a$:

$$1 = A(a-b)(a-c) + 0B + 0C$$

$$A = \frac{1}{(a-b)(a-c)}$$

If $x=b$:

$$1 = 0A + B(b-a)(b-c) + 0C$$

$$B = \frac{1}{(b-a)(b-c)}$$

If $x=c$:

$$1 = 0A + 0B + C(c-a)(c-b)$$

$$C = \frac{1}{(c-a)(c-b)}$$

So $\int \frac{1}{(x-a)(x-b)(x-c)}dx = \int \frac{1}{(a-b)(a-c)} \cdot \frac{1}{(x-a)}dx + \int \frac{1}{(b-a)(b-c)} \cdot \frac{1}{(x-b)}dx + \int \frac{1}{(c-a)(c-b)} \cdot \frac{1}{(x-c)}dx$

$$= \frac{\ln|x-a|}{(a-b)(a-c)} + \frac{\ln|x-b|}{(b-a)(b-c)} + \frac{\ln|x-c|}{(c-a)(c-b)} + C$$