Exam 1 Calc 2 9/10/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Set up** an integral for the present value of an income stream of $p(t) = 50000 - 5000t$ over the next ten years.

Present value $= \int_0^m P(t)e^{-rt}\,dt$

$$\int_0^{10} (50000 - 5000t)e^{-rt}\,dt$$

Good

2. Suppose that 6J of work are required to hold a spring stretched to a length of 60cm rather than its natural length of 50cm. How much work is required to stretch it from 50cm to 80cm?

$$6 = \int_0^{.1} Kx = \left[\tfrac{1}{2}Kx^2\right]_0^{.1} = \tfrac{1}{2}\cdot K\cdot\left(\tfrac{1}{10}\right)^2 = K\left(\tfrac{1}{200}\right)$$

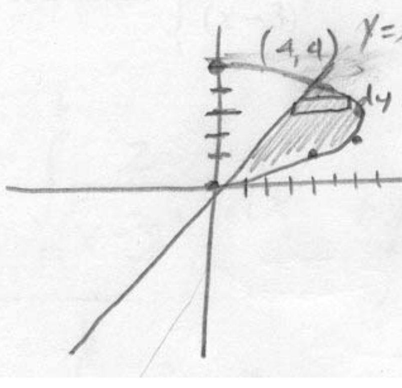
$$6 = \tfrac{1}{200}K \qquad K = 1200$$

$$\int_0^{.3} 1200x = \left[600x^2\right]_0^{.3} = 600\cdot\left(\tfrac{3}{10}\right)^2 = 600\left(\tfrac{9}{100}\right)$$

$$6\cdot 9 = 54\text{ J}$$

Great

3. **Set up** an integral for the area of the region between $x = 5y - y^2$ and $y = x$.



width of box $= \Delta y$

length $= (5y - y^2) - y$

Area of box $= (5y-y^2) - y\,\Delta y$

Area of region $= \int_0^4 (5y-y^2) - y\,dy$

Area of region: $\int_0^4 (5y - y^2) - y \quad dy$

Nice

4. **Set up** an integral for the length of the curve $y = x^2$ between the points (-1,1) and (3, 9).

$$AL = \int \sqrt{1 + (f(x)')^2}\, dx$$

$$y = x^2 \qquad \frac{dy}{dx} = 2x$$

$$\text{Arc length} = \int_{-1}^{3} \sqrt{1 + (2x)^2}\, dx$$

Great

5. Evaluate $\int_1^4 \frac{2}{(x-3)^2} dx$.

$$= \lim_{b \to 3} \int_1^b \frac{2}{(x-3)^2} dx + \lim_{a \to 3} \int_a^4 \frac{2}{(x-3)^2} dx$$

let $u = (x-3)$ $\frac{du}{dx} = 1$ $du = dx$

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$$= \lim_{b \to 3} \int_{x=1}^{x=b} \frac{2}{u^2} du + \lim_{a \to 3} \int_{x=a}^{x=4} \frac{2}{u^2} du$$

$$= \lim_{b \to 3} 2\int_{x=1}^{x=b} u^{-2} du + \lim_{a \to 3} 2\int_{x=a}^{x=4} u^{-2} du$$

$$= \lim_{b \to 3} 2(-u^{-1}) \Big|_{x=1}^{x=b} + \lim_{a \to 3} 2(-u^{-1}) \Big|_{x=a}^{x=4}$$

$$= \lim_{b \to 3} -\frac{2}{(x-3)} \Big|_1^b + \lim_{a \to 3} -\frac{2}{(x-3)} \Big|_a^4$$

$$= \lim_{b \to 3} -\frac{2}{b-3} - \left(-\frac{2}{(1-3)}\right) + \lim_{a \to 3} -\frac{2}{(4-3)} - \left(-\frac{2}{a-3}\right)$$

$$= -\frac{2}{0} - 1 - 2 + \left(\frac{2}{0}\right) \quad \boxed{\text{both diverge}}$$

well done

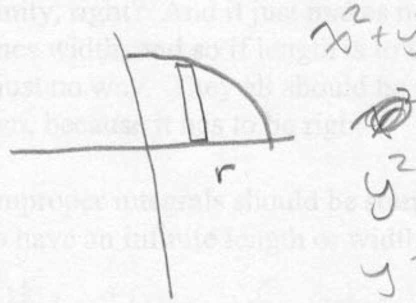
6. Suppose that within some Central American district the probability of a household having an income of d dollars/year is given by $p(d) = \dfrac{1}{2000} - \dfrac{d}{8000000}$ for values of d between 0 and the district's maximum household income. **Write an equation** which, when solved for b, gives the median household income in this district.

$$\boxed{\frac{1}{2} = \int_0^b \left(\frac{1}{2000} - \frac{d}{8000000}\right) dd}$$

Exactly.

7. **Set up** integrals for the x-coordinate of the center of mass of the first-quadrant portion of a circle with radius r.

$x^2 + y^2 = r^2$



$y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$

$$\boxed{\bar{x} = \frac{\int_0^r x\sqrt{r^2 - x^2}\, dx}{\int_0^r \sqrt{r^2 - x^2}\, dx}}$$

Excellent

8. Bunny is a calculus student at a large state university, and she's having some trouble. Bunny says "Ohmygod, this calculus stuff is sooooo freaky. I mean, there was this stuff in class today, like where you find the area of stuff that goes to infinity, right? And it just makes no sense at all, 'cause everybody knows you do area with length times width, and so if length is to infinity then automatically area is infinity too, right? So there's just no way. They all should be infinity, and then on the test I guess I'll just say that for all of them, because it has to be right.

Explain clearly to Bunny why the values of some improper integrals should be something other than infinity, even though the regions they pertain to have an infinite length or width.

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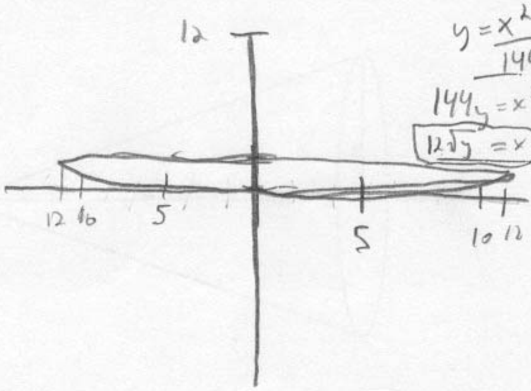
First of all Bunny, not all improper integrals have infinity in them. However, some improper integrals do have infinity in them but instead of always increasing the area, the area approaches a limit. Even though the length or width is going to infinity, the area under the curve is getting smaller all the time. This makes the total area reach a limit or real number.

The integrals that do not have infinity in them are much harder to find. They exist because one of the limits of the integral makes the denominator zero and you cannot divide by zero. For these neither the length or width is approaching infinity and the integrals need to be done as a limit. You need to try and figure out what the area beneath the curve looks like as it approaches a number or goes to infinity or negative infinity. Good

9. The horse that lives in the pasture next door to Jon's house is leaning over the fence and drinking from the birdbath in Jon's garden. The basin of the birdbath is shaped like the solid of revolution formed by taking the region above $y = x^2/144$ and below $y = 1$. If the basin starts out full of water, and the water all gets slurped up to the back of the horse's throat, 12 inches above the bottom of the birdbath, **set up** an integral for the amount of work the horse does in the process.



$y = \frac{x^2}{144}$

$144y = x^2$

$\boxed{12\sqrt{y} = x}$

radius of a slice: $\sqrt{y}$ ft

area of a slice: $\pi(\sqrt{y})^2 = \pi y$ ft^2

volume " ": $\pi y \cdot \Delta y$ ft^3

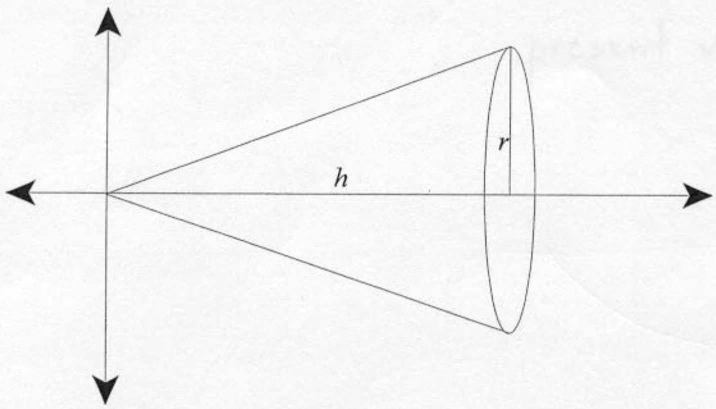
force " ": $\pi y \cdot \Delta y \cdot 62.4$ lbs

work " ": $62.4\pi y(1-y)\,\Delta y$ ft lbs

Total work = $\boxed{\int_0^{\frac{1}{12}} 62.4\pi y(1-y)\,dy}$

Wonderful!

10. A cone can be obtained by revolving the region below a certain line around the x axis, as suggested in the picture below. **Set up** an integral for the volume of a cone with height h and base radius r **and use it** to show that the volume of such a cone is $V = \frac{1}{3}\pi r^2 h$.



radius of slice $= \frac{r}{h}x$

area of slice $= \pi\left(\frac{r}{h}x\right)^2$

volume of slice $= \pi\left(\frac{r}{h}x\right)^2 \cdot \Delta h$

$$\boxed{\text{total volume} = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx}$$

$$V = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx$$

$u = \frac{r}{h}x$

$\frac{du}{dx} = \frac{r}{h}$

$du\,h = r\,dx$

$du\,\frac{h}{r} = dx$

$$V = \frac{h}{r}\pi\int_0^h u^2\,du$$

$$V = \frac{h}{r}\pi\left[\frac{u^3}{3}\Big|_0^h\right]$$

$$V = \frac{h}{r}\pi\left[\frac{\left(\frac{r}{h}x\right)^3}{3}\Big|_0^h\right]$$

$$V = \frac{h}{\cancel{r}}\pi\cdot\left[\frac{r^{\cancel{3}\,2}}{3}\right]$$

$$V = \frac{h}{3}\pi r^2$$

$$\boxed{V = \frac{1}{3}\pi r^2 h}$$

Beautiful Job!

Darn ~~Damn~~ right, didn't think I could do that.

Extra Credit (5 points possible):