Exam 3 Calc 2 11/5/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Find the sum** of the series $4-1+\frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\ldots$

$\text{Sum} = \frac{a}{1-r}$ — a = 1st term, r = ratio between terms

Ratio $= \frac{-1}{4} = -\frac{1}{4}$

$a=4$

$S = \frac{4}{1-(-\frac{1}{4})} = \frac{16}{5}$

Excellent

2. Write the series $-\frac{1}{3}+\frac{2}{5}-\frac{3}{7}+\frac{4}{9}-\frac{5}{11}+\ldots$ in sigma notation.

alternating series hence $(-1)^n$

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{2n+1}$$

Wonderful

always produces odd #s in denominator

b/c multiplying # by 2 = even #

but adding 1 always gives odd #

3. Find the Taylor series for $\sin x$ of degree 4 centered at $a = \pi/2$.

$$P_4(x) = f\left(\tfrac{\pi}{2}\right) + \cancel{f'\left(\tfrac{\pi}{2}\right)\left(x-\tfrac{\pi}{2}\right)} + \frac{f''\left(\frac{\pi}{2}\right)\cdot\left(x-\frac{\pi}{2}\right)^2}{2!} + \cancel{\frac{f'''\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)^3}{3!}} + \frac{f^{(4)}\left(\frac{\pi}{2}\right)\cdot\left(x-\frac{\pi}{2}\right)^4}{4!}$$

$f(x) = \sin x \qquad f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{2}\right) = 1$

Nice job!!

Polynomial $\quad 1 + \frac{(-1)\left(x-\frac{\pi}{2}\right)^2}{2} + \frac{\left(x-\frac{\pi}{2}\right)^4}{24}$

$$\boxed{1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2} + \frac{\left(x-\frac{\pi}{2}\right)^4}{24}}$$

4. Find the first 5 non-zero terms of the MacLaurin series for $f(x) = e^{(x^3)}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

So $e^{x^3} = 1 + x^{(3)} + \frac{(x^2)^3}{2!} + \frac{(x^3)^3}{3!} + \frac{(x^4)^3}{4!} + \frac{(x^5)^3}{5!}$

$$\boxed{P\ e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!}}$$

Well done!

5. Find the **radius of convergence** of the series $\sum_{n=0}^{\infty} \frac{(3x)^n}{2n+1}$.

$$\lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{(3x)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{3x\cancel{(3x)^n}}{2n+3} \cdot \frac{2n+1}{\cancel{(3x)^n}} \right|$$

$$3x \lim_{n\to\infty} \left| \frac{2n+1}{2n+3} \right|$$

$$\overset{L'H}{=} |3x| \lim_{n\to\infty} \left| \frac{2}{2} \right|$$

$$= (3x)|1|$$

$|3x| < 1$ b/c for Rat. test to work answer must be less than 1

$|x| < \frac{1}{3}$

$$\boxed{-\frac{1}{3} < x < \frac{1}{3}}$$

Excellent!

6. Use a series of degree at least 5 to **approximate ln 2.**

$\ln|1-x|$ is almost the antiderivative of $\frac{1}{1-x}$, whose series is $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$

$-\ln|1-x|$ is the antiderivative of $\frac{1}{1-x}$

so if each term is taken the antiderivative of and multiplied by -1, you will get a series for $\ln|1-x|$

so series approximation for $\ln|1-x| = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

so to find $\ln(2)$, plug (-1) in for x

$$\ln 2 = -(-1) - \left(\frac{1}{2}\right) - \left(-\frac{1}{3}\right) - \left(\frac{1}{4}\right) - \left(-\frac{1}{5}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

$$= \frac{60}{60} - \frac{30}{60} + \frac{20}{60} - \frac{15}{60} + \frac{12}{60} = \boxed{\frac{47}{60}}$$

Great Job!

7. Biff is a calculus student at a large state university, and he's having some trouble. Biff says "Man, this series stuff is killing me. Our professor gave us, like, this study sheet with a bunch of problems that might be on our test, like thirty of 'em, and he's gonna pick just some of 'em for the test. He said it's multiple choice, just say if it converges or if it diverges. So one of the problems was for the series one over x cubed, and I think he screwed up, 'cause that one doesn't converge or diverge, it's neither, 'cause the ratio test says one for it, so that means it doesn't converge or diverge, right? So I guess the exam really has to be with three choices, like 'converge', or 'diverge', or 'can't tell', right?"

Explain clearly to Biff whether what he's saying is true or false, and why.

Biff, the problem you are having is that you only tried one of the tests. If one of the tests tells you absolutely nothing, then try another. In this case I would try the p-series test. The p-series test says that when you have an something like $a_n = \frac{1}{x^p}$, if P is greater than 1 the series will converge. In your case, $P = 3\left(\frac{1}{x^3}\right)$ so the series you were given would converge. Just remember, just because one test tells you nothing, doesn't mean another test will do the same.

Exactly!

8. Determine whether $x = -5$ is included in the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{5^n \cdot n}$.

If $x=-5$:

$$\sum_{n=0}^{\infty} \frac{(-5)^n}{5^n \cdot n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

Which is the alternating harmonic series, which converges.

9. Determine whether the series $\sum_{n=0}^{\infty} \frac{2^n}{n!+1}$ converges or diverges. Be very clear in your justification.

~~Ratio test!~~ $\cancel{\frac{2^{n+1}}{(n+1)n!+1} \cdot \frac{n!+1}{2^n} = \frac{2(n!+1)}{(n+1)n!+1}}$

let's see if $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges. If it does $\sum_{n=0}^{\infty} \frac{2^n}{n!+1}$ is less than it, so it converges as well.

Ratio test! $\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)n!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0$ so $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges by the ratio test.

Comparison test! $\frac{2^n}{n!} > \frac{2^n}{n!+1}$ and $\frac{2^n}{n!}$ converges.

Therefore $\frac{2^n}{n!+1}$ converges as well, by the comparison test

Beautiful Job! Converges!

10. Show how a Taylor polynomial can be used to find $\lim_{x \to 0} \frac{\sin 3x}{x}$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

$$\sin(3x) = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}$$

$$\frac{\sin(3x)}{x} = 3 - \frac{3^3 x^2}{3!} + \frac{3^5 x^4}{5!} - \frac{3^7 x^6}{7!}$$

plug 0 in for x and you would get

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Exactly!