Exam 4 Calc 2 12/3/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Circle all of the candidates below which **are** differential equations:

a) $\frac{dy}{dx} = ky$

✓ is an equation
✓ has a derivative

b) $5x = 25$

c) $y = e^{kt}$

Excellent!

d) $y'' + 7y' + 12y = 0$

e) $0 = 1$

f) 42

g) zebras = mammals

2. Determine whether $y = x\,e^x$ is a solution to the differential equation $x\,y' = y + x\,y$.

$y' = xe^x + e^x \cdot 1$
Product rule

$x(xe^x + e^x) = xe^x + x(xe^x)$

$x^2e^x + xe^x = xe^x + x^2e^x$

Great!!

Yes, this is a solution because the left side matches the right side!

3. The differential equation $\frac{dm}{dt} = 0.1m + 200$ has general solution $m(t) = Ae^{0.1t} - 2000$. Find a particular solution satisfying the initial condition $m(0) = 1500$.

$m(0) = 1500 = Ae^{.1(0)} - 2000$

$1500 = A \cdot 1 - 2000$

$+2000 \qquad +2000$

$3500 = A$

Good

$$\boxed{m(t) = 3500e^{0.1t} - 2,000}$$

4. An object is placed in a 68° room. Write a differential equation for H, the temperature of the object at time t.

$\frac{dH}{dt} = K(A - H)$

Yep!

$$\boxed{\frac{dH}{dt} = K(68 - H)}$$

5. Find the general solution to the differential equation $y' = \sqrt{x}$.

$$\frac{dy}{dx} = \sqrt{x}$$

$$\int dy = \int \sqrt{x}\, dx$$

$$y = \int x^{1/2}\, dx$$

$$y = \frac{x^{3/2}}{3/2}$$

$$\boxed{y = \tfrac{2}{3} x^{3/2} + C}$$

Yup

6. Consider a pond that has an initial volume of 10,000 cubic meters. Suppose that at time $t = 0$, the water in the pond is clean and that the pond has two streams flowing into it, stream A and stream B, and one stream flowing out, stream C. Suppose 500 cubic meters per day of water flow into the pond from stream A, 750 cubic meters per day flow into the pond from stream B, and 1250 cubic meters flow out of the stream via stream C.

At time $t = 0$, the water flowing into the pond from stream A becomes contaminated with road salt at a concentration of 5 kilograms per 1000 cubic meters. Suppose water in the pond is well mixed so that the concentration of salt at any given time is constant. To make matters worse, suppose also that at time $t = 0$ someone begins dumping trash into the pond at a rate of 50 cubic meters per day. To adjust for the incoming trash, the rate that water flows out via stream C increases to 1300 cubic meters per day and the banks of the pond do not overflow.

This situation can be modeled by the differential equation $\frac{dS}{dt} = \frac{5}{2} - \frac{26S}{200-t}$ for values of t between 0 and 200. Solving this differential equation is beyond our techniques, but we can approximate. Use Euler's method with $\Delta t = 10$ and the initial condition $S(0) = 0$ to approximate the amount of salt in the pond after 20 days have passed.[1]

$S(0)=0$

$$\frac{dS}{dt} = \frac{5}{2} - \frac{26S}{200-t}$$

starting w/ $S(0)=0$

+ 25 after 10 days

$S(10)=25$

$$\frac{\Delta S}{\Delta t} = \frac{5}{2} - \frac{26(0)}{200-0}$$

$$\frac{\Delta S}{\Delta t} = \frac{5}{2}$$

$$\Delta S = \frac{5}{2}\,\Delta t$$

$$\Delta S = \frac{5}{2}(10) \qquad \Delta S = 25$$

$$\frac{\Delta S}{\Delta t} = \frac{5}{2} - \frac{26(25)}{200-10}$$

Excellent

$$\frac{\Delta S}{\Delta t} = -.921 \qquad \Delta S = -9.21$$

$$\Delta S = -.921\,\Delta t \qquad 25 - 9.21 = 15.78$$

$$\Delta S = -.921(10) \qquad \boxed{S(20) = 15.789\ \frac{kg}{1000m^3}}$$

[1] Shamelessly snatched from Blanchard-Devaney-Hall's *Differential Equations*, 3rd edition

7. The downward velocity of a falling object is modeled by the differential equation $\frac{dv}{dt} = 32 - 0.4v^2$. If $v(0) = 0$, the velocity will increase to a terminal velocity. The terminal velocity is an equilibrium solution where the upward air drag exactly cancels the downward gravitational force. Find the terminal velocity.[2]

equilibrium is when $\frac{dV}{dt} = 0$

or there is no change

$0 = 32 - 0.4v^2$

$.4v^2 = 32$

$v^2 = \frac{32}{.4}$

Excellent

$V = \sqrt{\left(\frac{32}{.4}\right)}$

$V = 8.94$

8. Biff is having some trouble with differential equations. He says "Dude, here's my thing with all this. So I guess there's lots of times when differential equations have, like, infinitely solutions, right? But so I was wondering, like, if there were maybe some differential equations that only have just one solution, or maybe just two or something, instead of having infinitely many. Can that happen?"

Explain to Biff, in terms he can understand, whether it can happen that a differential equation might have only one or two solutions, and why.

10

Outstanding answer.

Well Biff let's start out this way; it <u>is</u> possible for the <u>same</u> differential equation to have infinitely many solutions and only one solution. To identify which it can be first we need to learn some Calculus vocabulary. The first term is a <u>general</u> solution. A general solution is the answer to a differential equation but with no values assigned to the constants. This means you'll see letters like <u>C</u> or <u>K</u> in the answer. Since the letters are constants they can be infinitely many numbers in different equations. Since they can be infinitely many numbers there can be infinitely many solutions. The second term is a <u>particular</u> solution. A particular solution has values assigned to the letters in the solution like $C = 100$ and $k = 0.005$. When the constants have values assigned to them, that is the <u>only</u> solution. You can tell the difference between a general solution and a particular solution by the phrasing of the question. A particular solution will have statements like $y(0) = 10$ + $y(1) = 10$ in the question while a general solution won't.

9. Carbon-14 has a half-life of 5730 years. Write a differential equation for the breakdown of carbon-14 in an object, give a solution to this differential equation provided that the amount of carbon-14 present when $t = 0$ is given by A_0, and (to the nearest year) determine how long it takes until only 80% of the initial amount of carbon-14 in an object remains.

$$\boxed{\frac{dC_{14}}{dt} = K \cdot C_{14}}$$

$$\int \frac{1}{C_{14}}\, dC_{14} = \int K\, dt$$

$$\ln C_{14} = Kt + C$$

$$e^{Kt+C} = C_{14}$$

$$C_{14} = A_0 e^{Kt}$$

$$\frac{1}{2} A_0 = A_0 e^{5730K}$$

$$\frac{1}{2} = e^{5730K}$$

$$\ln \frac{1}{2} = 5730K$$

$$-.693147 = 5730K$$

$$K = -.00012097$$

$$C_{14} = A_0 e^{-.00012097t}$$

Wonderful

$$.8 A_0 = A_0 e^{-.00012097t}$$

$$.8 = e^{-.00012097t}$$

$$\ln .8 = -.00012097t$$

$t \approx 1845$ yrs until 80% of the original amount remains

10. Suppose that f(x) is a function for which $\int f(x)\,dx = F(x)$. Find a general solution to the differential equation $f'(x) = f(x)\,y + y$.

$$\frac{dy}{dx} = f(x)y + y$$

$$\frac{dy}{dx} = (f(x)+1)\,y$$

$$\int \frac{1}{y}\,dy = \int (f(x)+1)\,dx$$

$$= \int f(x)\,dx + \int 1\,dx$$

$$\ln|y| = F(x) + x + C$$

Brilliant!

$$\boxed{y = B\,e^{F(x)+x}}$$