Exam 1 Calc 3 9/24/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of f(x, y) with respect to *y*.

- The partial definition of the partial derivative of $f(x,y)$ with respect to y is given by,

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Yes

2. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^3 + y^3}$ does not exist. Be clear about the reason.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^3+y^3}$$

look along the line $y = x$:

$$\lim_{(x,x)\to(0,0)} \frac{x^2 \cdot x}{x^3+x^3} = \lim_{(x,x)\to(0,0)} \frac{x^3}{2x^3} = \boxed{\frac{1}{2}}$$

look along the line $y = 0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2 \cdot 0}{x^3+0^3} = \lim_{(x,0)\to(0,0)} \frac{0}{x^3} = \boxed{0}$$

Exactly

If the value of the function is different when approaching (0,0) from different places, then the limit does not exist.

3. Find the gradient of the function f(x, y) = (x + y)e^y.

$x + 2e^{\phi}$ $\qquad$ $xe^y + ye^y$

grad f(x,y) $\quad f_x(x,y) = e^y$

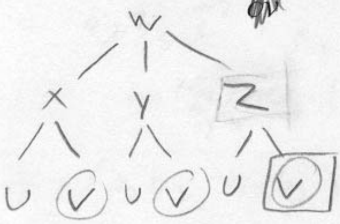
$f_y(x,y) = 1e^y + ye^y + xe^y$

$\therefore$ grad $(x,y) = e^y\vec{i} + (e^y + ye^y + xe^y)\vec{j}$

Great!

grad x·y $= f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$

4. If $w = f(x, y, z)$, $x = g_1(u, v)$, $y = g_2(u, v)$, and $z = g_3(u, v)$, state the appropriate form of the chain rule for $\frac{\partial z}{\partial v}$. Be sure to indicate clearly whether each derivative is a partial or regular.



$$\boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}}$$

all partial derivatives $\qquad$ Exactly

5. If $f(x, y) = \ln(ye^{xy})$, find the directional derivative of f in the direction of the vector $\vec{u} = \frac{(5\vec{i} - 12\vec{j})}{13}$.

$\vec{u} = \frac{5}{13}\vec{i} - \frac{12}{13}\vec{j}$ is a unit vector as magnitude $|\vec{u}| = 1$

$\|u\| = \sqrt{\left(\frac{5}{13}\right)^2 + \left(\frac{-12}{13}\right)^2} = \sqrt{\frac{25+144}{169}} = \sqrt{\frac{169}{169}} = 1$

$f_x = \ln(ye^{xy})$

$= \frac{1}{ye^{xy}}\, y \cdot e^{xy} \cdot y$

$= \frac{y^2 \cancel{e^{xy}}}{\cancel{y}\,\cancel{e^{xy}}}$

$= y$

$f_y = \ln(ye^{xy})$

$= \frac{1}{ye^{xy}} \cdot \left[e^{xy} + y \cdot e^{xy} \cdot x\right]$

$= \frac{1}{y} + x.$

Excellent

$\therefore$ Directional derivative $= \frac{5}{13}y + \left(\frac{-12}{13}\right)\left(\frac{1}{y} + x\right)$

6. Prove that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

let $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2 b_3 \vec{i} + a_1 b_2 \vec{k} + a_3 b_1 \vec{j} - \left(a_2 b_1 \vec{k} + a_3 b_2 \vec{i} + a_1 b_3 \vec{j} \right)$$

$$= \left[(a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k} \right]$$

if $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$, then $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \left[(a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k} \right] \cdot (a_1 \vec{i}, a_2 \vec{j}, a_3 \vec{k})$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1$$

now things cancel

Excellent Job

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$$= 0$$

because $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$, $(\vec{a} \times \vec{b})$ must be perp. to $\vec{a}$.

7. Biff is a calculus student at a large state university, and he's having some trouble. Biff says "Man, this stuff is hard. There was this question on our test about, like, if the level curves of this thing were all spaced out evenly, what did that mean, right? So I said, like, it meant the thing was a plane, 'cause the professor said something like that in class I think. But I got it back with 1 point, and it turns out this guy doesn't even grade on a curve. That curve was the only thing that got me through Calc 2, what the heck am I going to do now?"

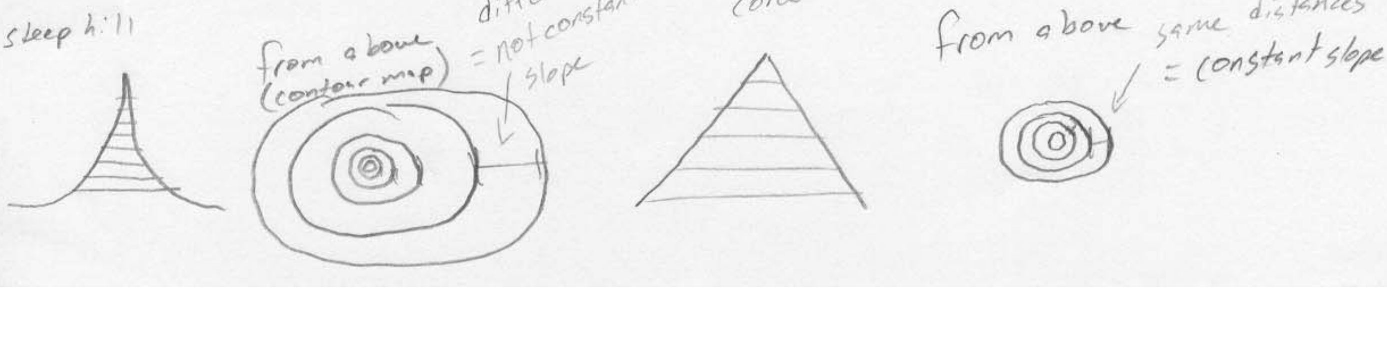
Explain to Biff what conclusion you can draw if you know a surface's level curves are evenly spaced, and why.

Remember geography, Biff? Math isnt your forte so lets think about geography. and contour maps.

The level curves are like the contours on a contour map. They connect all the x,y pairs that share the same z-coordinate. If they are all the same distance apart, then the slope is constant in that direction. If the lines weren't evenly spaced then the slope is not constant, it's changing. Even though a plane does have evenly spaced contours, it is <u>not</u> the <u>only</u> shape that does. Look at a <u>cone</u> for instance.

10

Wonderful answer.



8. Write an equation for the plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ passing through the origin. Considering the plane as a function of x and y, in which direction will its directional derivative be greatest, and how large will the directional derivative be in that direction?

$0 = ax + by + cz$ is the equation of plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$

$z = -\frac{a}{c}x - \frac{b}{c}y.$

Now. $\text{grad}(x, y) = -\frac{a}{c}\vec{i} - \frac{b}{c}\vec{j}$

So the directional derivative will be greatest at direction $= -\frac{a}{c}\vec{i} - \frac{b}{c}\vec{j}$

and the magnitude will be

$$= \sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}}.$$

Perfect.

$$= \frac{\sqrt{a^2+b^2}}{c}.$$

9. Find all points on the surface $z = x^2 + y^2 + 2$ whose tangent planes pass through the origin.

$z_x = 2x \qquad z_y = 2y$

The plane tangent to this surface at the point (a, b, a^2+b^2+2) is:

$$z - (a^2+b^2+2) = 2(a)(x-a) + 2(b)(y-b)$$

or

$$z - a^2 - b^2 - 2 = 2ax - 2a^2 + 2by - 2b^2$$

For this to pass through the origin means $(0,0,0)$ works for (x,y,z):

$$(0) - a^2 - b^2 - 2 = 2a(0) - 2a^2 + 2b(0) - 2b^2$$

$$-a^2 - b^2 - 2 = -2a^2 - 2b^2$$

$$a^2 + b^2 = 2$$

So these points whose tangent planes pass through the origin lie on a circle centered at the origin with radius $\sqrt{2}$ and height 4

10. Find a function f(x, y, z) for which $f_x = 2x - 3z$, $f_y = -6y^2z$, and $f_z = 3x - 2y^3 + 5$, or explain why one cannot exist.

x^2, $3xz$, $5z$, $-2y^3z$

$f(x,y,z) = x^2 + 3xz - 2y^3z + 5z$

$f_x(x,y,z) = 2x + 3z$

$f_y(x,y,z) = -6y^2z$

$f_z(x,y,z) = 3x - 2y^3 + 5$

No function can exist with these partial derivatives because of the discrepancy in the sign on $3z$ in f_x and $3x$ in f_z. These terms in the partials must have come from the same terms in the functions so the sign would have to be the same.

Excellent Reasoning.