Exam 3 Calc 3 12/3/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

$$\boxed{\int_C \langle P,Q\rangle \cdot d\vec{r} = \iint_D \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dA}$$

1. Compute the divergence of the vector field $\vec{F} = \boxed{2xy}\,\vec{i} + \boxed{y^2}\,\vec{j} - \boxed{3}\,\vec{k}$.

$$\text{divergence} = \frac{\partial(2xy)}{\partial x} + \frac{\partial(y^2)}{\partial y} + \frac{\partial(-3)}{\partial z}$$

$$\text{divergence} = 2y + 2y + 0$$

$$\text{divergence} = 4y$$

Good

2. Compute the curl of the vector field $\vec{F} = 2xy\,\vec{i} + y^2\,\vec{j} - 3\,\vec{k}$.

$$\vec{F} = \langle 2xy,\ y^2,\ -3 \rangle$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & y^2 & -3 \end{vmatrix} = (0\vec{i} + 0\vec{j} + 0\vec{k}) - (2x\vec{k} + 0\vec{j} + 0\vec{i})$$

$$= -2x\vec{k}$$

$$\text{curl}\,\vec{F} = \langle 0,\ 0,\ -2x \rangle$$

Great

3. Evaluate $\int_C \langle -\sin y \sin x, \cos y \cos x \rangle \cdot d\vec{r}$ where C is the line segment from $(\pi/2, 0)$ to $(0, \pi/2)$

f_x f_y

$\vec{F} = \langle -\sin y \sin x, \cos y \cos x \rangle$

$-\cos y \sin x \underset{?}{=} -\cos y \sin x$ ∴ mixed partials = so $\vec{F}$ is conservative

~~cosy~~ $+\sin y \cos x$ = potential function

$\sin y \cos x \Big|_{\frac{\pi}{2}, 0}^{0, \frac{\pi}{2}}$ $\left(\sin\left(\frac{\pi}{2}\right)\cos(0)\right) - \left(\sin 0 \cos \frac{\pi}{2}\right)$

$= 1$ 0

Excellent

$\int_C \vec{F} \cdot d\vec{r} = 1$ using Fundamental Theorem of Line Integrals

4. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle -y, x \rangle$ and C is the right half of a circle of radius 5 centered at the origin, traversed counterclockwise, then followed by the line segment from (0, 5) to (0, -5).

using Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint 1 + 1 \, dA$$

$$= 2 \iint dA$$

$$= 25\pi$$

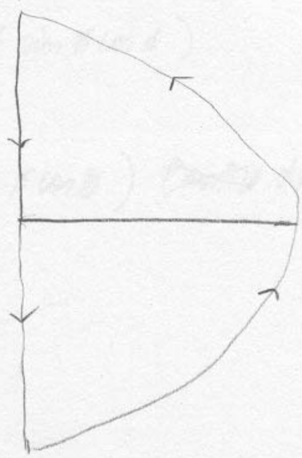
Nice!

Area $= \pi r^2$

$= \frac{\pi 25}{2}$

$\left(\text{Area} = \frac{\pi r^2}{2} = \frac{25\pi}{2}\right)$

(0,5)



(0,-5)

5. Let a, b, and c be constants. Compute $\iint_S \langle a, b, c \rangle \cdot d\mathbf{S}$ for S the surface of a sphere with radius 3 centered at the origin.

A sphere is a closed surface so use Divergence Theorem.

partials of $\langle a, b, c \rangle = \langle 0 + 0 + 0 \rangle = 0$

$\iiint 0 \, dV = \boxed{0}$

Nice Job!

6. If $\vec{F}$ is any vector field whose components have continuous second partial derivatives, show that div curl $\vec{F} = 0$.

$$\operatorname{div}(\operatorname{curl}\vec{F}) = 0$$

$$\vec{F} = \langle P, Q, R \rangle$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left(R_y\vec{i} + P_z\vec{j} + Q_x\vec{k}\right) - \left(P_y\vec{k} + R_x\vec{j} + Q_z\vec{i}\right)$$

$$= \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\operatorname{div}(\operatorname{curl}\vec{F}) = (R_y - Q_z)\,\partial/\partial x + (P_z - R_x)\,\partial/\partial y + (Q_x - P_y)\,\partial/\partial z$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$= 0 \checkmark$$

Well done!

Because as long as the 2nd partial derivatives are continuous, Clairaut's Theorem says that mixed partials are equal.

7. Biff is a calculus student at a large state university, and he's having some trouble. Biff says "Dude, this is crazy. Our professor is psychotic. He gave us this test that was so long I don't think anybody could do it in an hour. But one of the problems was totally beyond all the others, 'cause it had, like, these five different surfaces, like a cone, a paraboloid, the top half of a sphere, the bottom half of a sphere, and something else I can't even remember, and we were supposed to show their flux integrals were all equal. I mean, just doing five flux integrals would take anybody more than an hour! But to top it off he had the vector field was one where we had to do its curl too. The man has totally lost his mind. A bunch of us were talking about, like, calling the psych department and having them study him or something."

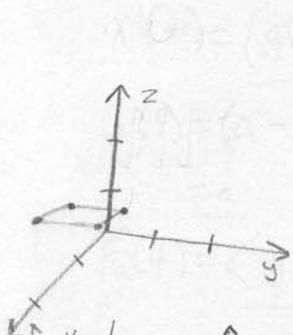
Explain to Biff, in terms he can understand, how the professor might have actually intended the problem mentioned to be possible.

OK Biff,

I think your professor is wanting you to do Stokes Theorm. In Stokes Theorm, It's $\iint_S \text{curl}\,\vec{F}\, dS = \int \vec{F} \cdot d\vec{r}$. Stokes Theorm gives you the chance to find line integrals instead of using surface integrals. He doesn't want you to find the curl, the curl $\vec{F}$ is the notation to use Stokes Theorm. So it's not as hard as you think. Stokes show us that only boundardies matter! W/ flux integrals, and Stokes Theorm, you can see how much fluxuation through the boundaries.

Excellent.

8. Compute $\int_S \langle x^2, 2y^2, 3z^2 \rangle \cdot d\vec{A}$ where S is the square with vertices (1,0,1), (2,0,1), (2,1,1), and (1,1,1).



$x = 1 \quad y = 0$

$x = 2 \quad y = 1 \quad z = 1$

$x(u,v) = u$

$y(u,v) = v$

$z(u,v) = 1$

$\vec{F}(u,v) = \langle u, v, 1 \rangle$

for $1 \leq u \leq 2$

$0 \leq v \leq 1$

$\vec{F}(\vec{r}(u,v)) = \langle u^2, 2v^2, 3(1)^2 \rangle$

$= \langle u^2, 2v^2, 3 \rangle$

$\vec{r}_u = \langle 1, 0, 0 \rangle \quad \vec{r}_v = \langle 0, 1, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0\vec{i} + 0\vec{j} + \vec{k}) - (0\vec{k} + 0\vec{j} + 0\vec{i})$$

$= \vec{k}$

$= \langle 0, 0, 1 \rangle$

$\iint_S \langle u^2, 2v^2, 3 \rangle \cdot \langle 0, 0, 1 \rangle dS$

$\iint_S 3\, dS \longrightarrow \int_1^2 \int_0^1 3\, dv\, du$

$\int_1^2 3v \Big|_0^1 du$

$\int_1^2 3\, du$

$3u \Big|_1^2$

$3(2) - 3(1)$

$6 - 3$

3

Great Job

9. Let C be a line segment from the point $(g_3(a), a)$ to the point $(g_1(a), a)$, and let $\vec{F}(x,y) = \langle 0, Q(x,y) \rangle$ for some function $Q(x, y)$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

$x(t) = (g_1(a) - g_3(a))t + g_3(a)$

$y(t) = (a - a)t + a$

$= a$

$\vec{F}(t) = \langle g_1(a)t - g_3(a)t + g_3(a), a \rangle$

$\vec{F}(\vec{r}(t)) = \langle 0, Q(g_1(a)t - g_3(a)t + g_3(a), a) \rangle$

$\vec{r}'(t) = \langle g_1(a) - g_3(a), 0 \rangle$

$\int_C \langle 0, Q(g_1(a)t - g_3(a)t + g_3(a), a) \rangle \cdot \langle g_1(a) - g_3(a), 0 \rangle \, d\vec{r}$

$\int_C 0 \, d\vec{r} = 0$

well done.

10. Suppose that C_1 is a line segment beginning at (x_1, y_1) and ending at (x_2, y_2), and that $\int_{C_1} \langle 4x, 2y \rangle \cdot d\vec{r} = 1$. How many paths C_n can exist which also begin at (x_1, y_1) and end at (x_2, y_2), but for which $\int_{C_n} \langle 4x, 2y \rangle \cdot d\vec{r} = 2$? Why?

$(2x^2 + y^2)$ is a potential function for $\vec{F}$.

this means the value of $\int_{C_n} \langle 4x, 2y \rangle \cdot d\vec{r}$ is independent of the path. $\int_{C} \langle 4x, 2y \rangle \cdot d\vec{r}$ has an infinite number of paths that have a value of 1, but since the value is independent of the path, the value will always equal 1, as long as you start at (x_1, y_1) and end at (x_2, y_2).

It can never equal 2, so the answer...

Wonderful

$\boxed{\text{0 paths } C_n \text{ exist which begin at } (x_1, y_1) + \text{end at } (x_2, y_2) \text{ for which } \int_{C_n} \langle 4x, 2y \rangle \cdot d\vec{r} = 2}$

Extra Credit (5 points possible):