Exam 1   Real Analysis 1   10/1/2004

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of a sequence.

2. State the definition of divergence of a sequence to $+\infty$.

3. Give an example of a sequence which is bounded but not convergent.

4. State the Bolzano-Weierstrass Theorem.

5. Prove that for any real numbers a and b, $|a - b| \leq |a| + |b|$.

6. Prove that the sequence $\left\{ \frac{n}{n+1} \right\}$ is convergent.

7. Using some or all of the axioms:

(A1) (*Closure*) $a + b$, $a \cdot b \in \mathbb{R}$ for any a, $b \in \mathbb{R}$. Also, if a, b, c, $d \in \mathbb{R}$ with $a = b$ and $c = d$, then $a + c = b + d$ and $a \cdot c = b \cdot d$.

(A2) (*Commutative*) $a + b = b + a$ and $a \cdot b = b \cdot a$ for any a, $b \in \mathbb{R}$.

(A3) (*Associative*) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any a, b, c, $\in \mathbb{R}$.

(A4) (*Additive identity*) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a + 0 = a$ for any $a \in \mathbb{R}$.

(A5) (*Additive inverse*) For each $a \in \mathbb{R}$, there exists an element $-a$ in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (*Multiplicative identity*) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (*Multiplicative inverse*) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (*Distributive*) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for any a, b, $c \in \mathbb{R}$.

(A9) (*Trichotomy*) For a, $b \in \mathbb{R}$, exactly one of the following is true: $a = b$, $a < b$, or $a > b$.

(A10) (*Transitive*) For a, $b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For a, b, $c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For a, b, $c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if a, b, c, $d \in \mathbb{R}^+$, with $a < b$ and $c < d$, then $ac < bd$. Be explicit about which axioms you use.

8. State and prove the Monotone Convergence Theorem.

9. Prove that if $x \in (0,1)$ is a fixed real number, then $0 < x^n < 1$ for all $n \in \mathbb{N}$.

10. Prove that if $\{a_n\}$ converges to A and $c \in \mathbb{R}$ then $\{c \cdot a_n\}$ converges.

Others receiving votes: By some combination of miracles, Biff has survived his misadventures in Calculus and is now a Real Analysis student. He's having some trouble with limits of sequences. Biff says "Dude, it all seems so complicated. Why can't they just say the limit is the number it gets closer and closer to? Why all the crap with epsilons and stars and all that?"

Give Biff at least one good, clear reason that just saying "closer and closer" isn't adequate to define convergence.