Exam 1 Real Analysis 1 10/1/2004

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of a sequence.

A sequence $\{a_n\}$ is a real valued function whose domain is $\mathbb{N}$.

Yes

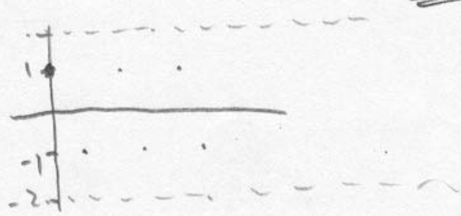
2. State the definition of divergence of a sequence to $+\infty$.

A sequence $\{a_n\}$ diverges to $+\infty$ iff for any $M \in \mathbb{N}$ there exists an $n^* \in \mathbb{N}$ such that $a_n > M$ for any $n > n^*$.

Exactly.

3. Give an example of a sequence which is bounded but not convergent.

The sequence $\{(-1)^n\}$ is bounded, but not convergent. $M = 2$ is a bound for it. Yes!



4. State the Bolzano-Weierstrass Theorem.

A bounded sequence must have at least one convergent subsequence.

Yes

5. Prove that for any real numbers a and b, $|a - b| \leq |a| + |b|$.

<u>Proof</u>: Well, $|a-b| = |a+(-b)| \leq |a| + |-b| = |a| + |b|$. This is just a simple variation of the triangle inequality.

So, from above $|a-b| \leq |a| + |b|$. $\square$

Exactly.

6. Prove that the sequence $\left\{\dfrac{n}{n+1}\right\}$ is convergent.

Scratch:

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon$$

$$\left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| < \varepsilon$$

$$\left|\frac{-1}{n+1}\right| < \varepsilon$$

$$\frac{1}{n+1} < \varepsilon$$

$$\frac{1}{\varepsilon} < n+1$$

$$\frac{1}{\varepsilon} - 1 < n$$

Well, let $\varepsilon > 0$ be given. Then let $n^* = \frac{1}{\varepsilon} - 1$ (or the next larger natural number). Then for $n > n^* = \frac{1}{\varepsilon} - 1$ we have

$$\frac{1}{\varepsilon} < n+1$$

or, since $\varepsilon > 0$ and $n+1 > 0$,

$$\frac{1}{n+1} < \varepsilon$$

but since $\left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$ this gives

$$\left|\frac{-1}{n+1}\right| < \varepsilon$$

or

$$\left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| < \varepsilon$$

which is just

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon$$

So when $n > n^*$, $|a_n - 1| < \varepsilon$, and thus a_n converges (to 1). $\square$

7. Using some or all of the axioms:

(A1) (*Closure*) $a + b$, $a \cdot b \in \mathbb{R}$ for any $a, b \in \mathbb{R}$. Also, if $a, b, c, d \in \mathbb{R}$ with $a = b$ and $c = d$, then $a + c = b + d$ and $a \cdot c = b \cdot d$.

(A2) (*Commutative*) $a + b = b + a$ and $a \cdot b = b \cdot a$ for any $a, b \in \mathbb{R}$.

(A3) (*Associative*) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c, \in \mathbb{R}$.

(A4) (*Additive identity*) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a + 0 = a$ for any $a \in \mathbb{R}$.

(A5) (*Additive inverse*) For each $a \in \mathbb{R}$, there exists an element $-a$ in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (*Multiplicative identity*) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (*Multiplicative inverse*) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (*Distributive*) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for any $a, b, c \in \mathbb{R}$.

(A9) (*Trichotomy*) For $a, b \in \mathbb{R}$, exactly one of the following is true: $a = b$, $a < b$, or $a > b$.

(A10) (*Transitive*) For $a, b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For $a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$, then $ac < bd$. Be explicit about which axioms you use. $a, b, c, d > 0$

Given $a < b$,

$\underline{ac < bc}$ by $\underline{A12}$, b/c $c > 0$

Also, given $c < d$,

$\underline{bc < bd}$ by $\underline{A12}$, b/c $b > 0$

Since $ac < bc$ and $bc < bd$,

$\underline{ac < bd}$, $\underline{\text{by A10.}}$ $\square$

Excellent

Great

8. State and prove the Monotone Convergence Theorem.

Def: A sequence $\{a_n\}$ that is bounded & monotone is convergent.

pf: Well, let's consider the case where $\{a_n\}$ is increasing & bounded. Look at the set $S=\{a_n \mid n\in\mathbb{N}\}$. Since $\{a_n\}$ is bounded, so is the set S. Since 'S' is bounded, it has a least upper bound, call it 'L'. We can say $a_n < L \ \forall n\in\mathbb{N}$ b/c 'L' is the LUB. Let $\varepsilon>0$ be given, & $L+\varepsilon > L$ by Axiom. By Axiom we can say $a_n < L < L+\varepsilon$. Thus $a_n < L+\varepsilon$. $\exists\, n^*\in\mathbb{N} \ni a_{n^*} > L-\varepsilon$ since 'L' is a LUB. And since $\{a_n\}$ is increasing $a_n > a_{n^*} \ \forall n > n^*$. Thus $a_n > a_{n^*} > L-\varepsilon$ by Axiom. With what we have from, we now have $L-\varepsilon < a_n < L+\varepsilon \ \forall n > n^*$.

$$
\begin{aligned}
L-\varepsilon &< a_n < L+\varepsilon \\
-L \quad & \quad -L \quad -L
\end{aligned}
$$

$-\varepsilon < a_n - L < \varepsilon$ & we know $|a|<b$ iff $-b<a<b$, so

$\boxed{|a_n - L| < \varepsilon.}$ Thus, $\{a_n\}$ converges & this case is proven. The other cases follow similarly. $\square$

Very nice job!

9. Prove that if $x \in (0,1)$ is a fixed real number, then $0 < x^n < 1$ for all $n \in \mathbb{N}$.

We are given $0 < x < 1$ and we want to show $0 < x^n < 1$. Well, lets try using induction!

1st lets see if it works for $n=1$.

$0 < x^1 < 1$ so $0 < x < 1$. Since this is our hypothesis we know it is true. ✓

2nd Now we assume it works for some $n=k$.

$0 < x^k < 1$.

Nice Job!

3rd Lets see if it works for some $n=k+1$

$0 < x^{k+1} < 1$. Lets break this up...

$0 < x^{k+1}$ is the same as $0 < x^k \cdot x$. Since I know $0 < x$ from my hypothesis, multiplying by x^k on both sides (A12) yields $0 \cdot x^k < x \cdot x^k$ so $0 < x^{k+1}$ as desired. ✓

Now $x^{k+1} < 1$. Well this is the same as $x^k \cdot x < 1$. Well from our hypothesis we know $x^k < 1$. multiplying by x we get $x^k \cdot x < x$ so $x^{k+1} < x$. From our hypothesis we know $x < 1$ $\therefore$ $x^{k+1} < x < 1$ and $x^{k+1} < 1$ as desired. ✓

Thus the inequality holds for all $n \in \mathbb{N}$

10. Prove that if $\{a_n\}$ converges to A and $c \in \mathbb{R}$ then $\{c \cdot a_n\}$ converges.

Proof: Well, let some $\epsilon > 0$ be given. Then $\exists$ an $n_1 \in \mathbb{N} \ni n > n_1 \Rightarrow |a_n - A| < \frac{\epsilon}{|c|}$ for some $c \in \mathbb{R}$. Then we have $|c| \cdot |a_n - A| < |c| \cdot \frac{\epsilon}{|c|}$ or $|c| \cdot |a_n - A| < \epsilon$ and by the prop: $|a| \cdot |b| = |ab|$ we can say $|(a_n - A)(c)| < \epsilon$ or $|c \cdot a_n - c \cdot A| < \epsilon$ $\therefore$ $\{c \cdot a_n\}$ converges by the def. of convergence.

Beautiful.