Each problem is worth 10 points. Show adequate justification for full credit. Don’t panic.

1. State the definition of a sequence.

   A sequence $\{a_n\}$ is a real valued function whose domain is $\mathbb{N}$.

2. State the definition of divergence of a sequence to $+\infty$.

   A sequence $\{a_n\}$ diverges to $+\infty$ iff for any $m \in \mathbb{N}$ there exists an $n^* \in \mathbb{N}$ such that $a_n > M$ for any $n > n^*$.

3. Give an example of a sequence which is bounded but not convergent.

   The sequence $\{(1/n^2)\}$ is bounded, but not convergent. $M = 2$ is a bound for it.
4. State the Bolzano-Weierstrass Theorem.

A bounded sequence must have at least one convergent subsequence.

Proof: Well, \(|a-b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|\). This is just a simple variation of the triangle inequality.

So, from above \(|a-b| \leq |a| + |b|\). \(\square\)

Exactly.
6. Prove that the sequence \( \left\{ \frac{n}{n+1} \right\} \) is convergent.

Well, let \( \varepsilon > 0 \) be given. Then let \( n^* = \frac{1}{\varepsilon} - 1 \) (or the next larger natural number). Then for \( n > n^* = \frac{1}{\varepsilon} - 1 \) we have

\[
\frac{1}{\varepsilon} < n + 1
\]

or, since \( \varepsilon > 0 \) and \( n + 1 > 0 \),

\[
\frac{1}{n + 1} < \varepsilon
\]

but since \( \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} \) this gives

\[
\frac{1}{n + 1} < \varepsilon
\]

or

\[
\left| \frac{n}{n + 1} - \frac{n + 1}{n + 1} \right| < \varepsilon
\]

which is just

\[
\left| \frac{n}{n + 1} - 1 \right| < \varepsilon
\]

So when \( n > n^* \), \( |a_n - 1| < \varepsilon \), and thus an converges \( \rightarrow 1 \). \( \square \)
7. Using some or all of the axioms:

(A1) (Closure) \( a + b, a \cdot b \in \mathbb{R} \) for any \( a, b \in \mathbb{R} \). Also, if \( a, b, c, d \in \mathbb{R} \) with \( a = b \) and \( c = d \), then \( a + c = b + d \) and \( a \cdot c = b \cdot d \).

(A2) (Commutative) \( a + b = b + a \) and \( a \cdot b = b \cdot a \) for any \( a, b \in \mathbb{R} \).

(A3) (Associative) \( (a + b) + c = a + (b + c) \) and \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \) for any \( a, b, c \in \mathbb{R} \).

(A4) (Additive identity) There exists a zero element in \( \mathbb{R} \), denoted by 0, such that \( a + 0 = a \) for any \( a \in \mathbb{R} \).

(A5) (Additive inverse) For each \( a \in \mathbb{R} \), there exists an element \(-a\) in \( \mathbb{R} \), such that \( a + (-a) = 0 \).

(A6) (Multiplicative identity) There exists an element in \( \mathbb{R} \), which we denote by 1, such that \( a \cdot 1 = a \) for any \( a \in \mathbb{R} \).

(A7) (Multiplicative inverse) For each \( a \in \mathbb{R} \) with \( a \neq 0 \), there exists an element in \( \mathbb{R} \) denoted by \( \frac{1}{a} \) or \( a^{-1} \), such that \( a \cdot a^{-1} = 1 \).

(A8) (Distributive) \( a \cdot (b + c) = (a \cdot b) + (a \cdot c) \) for any \( a, b, c \in \mathbb{R} \).

(A9) (Trichotomy) For \( a, b \in \mathbb{R} \), exactly one of the following is true: \( a = b, a < b, \) or \( a > b \).

(A10) (Transitive) For \( a, b, c \in \mathbb{R} \), if \( a < b \) and \( b < c \), then \( a < c \).

(A11) For \( a, b, c \in \mathbb{R} \), if \( a < b \), then \( a + c < b + c \).

(A12) For \( a, b, c \in \mathbb{R} \), if \( a < b \) and \( c > 0 \), then \( ac < bc \).

Prove that if \( a, b, c, d \in \mathbb{R} \), with \( a < b \) and \( c < d \), then \( ac < bd \). Be explicit about which axioms you use.

Given \( a < b \),

\[
\frac{ac}{bc} < \frac{bd}{bd} \quad \text{by} \quad \text{A12}, \quad \frac{bc}{bc} > 0
\]

Also, given \( c < d \),

\[
\frac{bc}{bd} < \frac{bd}{bd} \quad \text{by} \quad \text{A12}, \quad \frac{bc}{bd} > 0
\]

Since \( ac < bc \) and \( bc < bd \),

\[
ac < bd, \quad \text{by} \quad \text{A10}. \quad \square
\]
8. State and prove the Monotone Convergence Theorem.

Def: A sequence \( \{a_n\} \) that is bounded and monotone is convergent.

Prf: Well, let's consider the case where \( \{a_n\} \) is increasing and bounded.

Look at the set \( S = \{a_n \mid n \in \mathbb{N}\} \). Since \( \{a_n\} \) is bounded, so is the set \( S \).

Since \( S \) is bounded, it has a least upper bound, call it \( L \).

Since \( S \) is bounded, it has an upper bound, call it \( L \).

We can say \( a_n \leq L \) for all \( n \in \mathbb{N} \) b/c \( L \) is the LUB. Let \( \varepsilon > 0 \) be given, and \( L + \varepsilon > L \) by Axiom. By Axiom, we can say \( a_n \leq L + \varepsilon \).

Thus \( a_n \leq L + \varepsilon \). Since \( L + \varepsilon \) is a LUB, \( a_n \geq L - \varepsilon \).

And since \( \{a_n\} \) is increasing, \( a_{n+1} \geq a_n \) for all \( n \in \mathbb{N} \).

Thus \( a_n \geq a_n \).

And with what we have proven, we now have \( L - \varepsilon < a_n \leq L + \varepsilon \).

Thus \( |a_n - L| < \varepsilon \).

Thus, \( \{a_n\} \) converges and this case is proven. The other cases follow similarly. \( \square \)

Very nice job!
9. Prove that if $x \in (0,1)$ is a fixed real number, then $0 < x^n < 1$ for all $n \in \mathbb{N}$.

We are given $0 < x < 1$ and we want to show $0 < x^n < 1$. Well, let's try using induction!

1st: Let's see if it works for $n=1$.

$0 < x < 1$ so $0 < x^1$. Since this is our hypothesis we know it is true. ✅

2nd: Now we assume it works for some $n=k$.

$0 < x^k < 1$.

Nice job!

3rd: Let's see if it works for some $n=k+1$.

$0 < x^{k+1}$.

Let's break this up...

$0 < x^{k+1}$ is the same as $0 < x^k \cdot x$. Since we know $0 < x$ from my hypothesis, multiplying by $x$ on both sides yields $0 < x^k \cdot x^k$ so $0 < x^{k+1}$ as desired. ✅

Now $x^{k+1} < 1$. Well this is the same as $x^k \cdot x < 1$. Well from our hypothesis we know $x^k < 1$. Multiplying by $x$ we get $x^k \cdot x < x$. From our hypothesis we know $x < 1$ so $x^{k+1} < x$. From our hypothesis we know $x < 1$ and $x^{k+1} < 1$ as desired. ✅

Thus the inequality holds for all $n \in \mathbb{N}$.
10. Prove that if \( \{a_n\} \) converges to \( A \) and \( c \in \mathbb{R} \) then \( \{c \cdot a_n\} \) converges.

Proof: Well, let some \( \varepsilon > 0 \) be given. Then \( \exists \, n_0, N \in \mathbb{N} \, \forall \, n > N_1 \Rightarrow |a_n - A| < \frac{\varepsilon}{|c|} \) for some \( c \in \mathbb{R} \). Then we have \( |c| \cdot |a_n - A| < |c| \cdot \frac{\varepsilon}{|c|} \) or \( |c| \cdot |a_n - A| < \varepsilon \) and by the prop: \( |a| \cdot |b| = |a||b| \) we can say \( |(c \cdot a_n)(c)| < \varepsilon \) or \( |c \cdot a_n - c \cdot A| < \varepsilon \). \( \{c \cdot a_n\} \) converges by the def. of convergence.

Beautiful.