1.) Prove $\lim_{x \to a} x^3 = a^3$.

Want:
$|x^3 - a^3| < \varepsilon$

if $x > a$:
$|x^3 - a^3| = x^3 - a^3$
$x^3 - a^3 < \varepsilon$
$x^3 < \varepsilon + a^3$
$x < \sqrt[3]{\varepsilon + a^3}$

Well, let $\varepsilon > 0$ be given.

There exists $\delta = \begin{cases} \sqrt[3]{\varepsilon + a^3} - a & \text{if } x > a \\ \sqrt[3]{\varepsilon - a^3} + a & \text{if } x < a \end{cases}$

if $x > a$: $0 < |x-a| < \sqrt[3]{\varepsilon + a^3} - a$

since $x > a$, $|x-a| > 0$: $x - a < \sqrt[3]{\varepsilon + a^3} - a$

adding a: $x < \sqrt[3]{\varepsilon + a^3}$

cubing: $x^3 < \varepsilon + a^3$

subtracting a^3: $x^3 - a^3 < \varepsilon$

since $x > a$, $x^3 > a^3$, so $|x^3 - a^3| = x^3 - a^3$: $|x^3 - a^3| < \varepsilon$. ✓

if $x < a$: $0 < |x-a| < \sqrt[3]{\varepsilon - a^3} + a$

since $x < a$, $|x-a| < 0$: $-x + a < \sqrt[3]{\varepsilon - a^3} + a$
$|x-a| = -(x-a) = -x + a$:

subtracting a: $-x < \sqrt[3]{\varepsilon - a^3}$

cubing: $-x^3 < \varepsilon - a^3$

adding a^3: $-x^3 + a^3 < \varepsilon$

Since $x < a$, $|x^3 - a^3| = -(x^3 - a^3) = -x^3 + a^3$: $|x^3 - a^3| < \varepsilon$. ✓

Clever!

So you would use $\delta = \min\{\sqrt[3]{\varepsilon + a^3} - a, \sqrt[3]{\varepsilon - a^3} + a\}$.

These are the only to ways by trichotomy ($x < a$, $x > a$, $x = a$) and we don't have to worry about when $x = a$ because $0 < |x-a| < \delta$. $\square$

② $\lim_{x \to 0} \cos(1/x)$

No, Pick a sequence $\{x_n\}$ of distinct values that converges to $x=0$ but for which $f(x_n)$ diverges.

Let, $x_n = \frac{1}{n\pi}$, which converges to $x=0$, but

$$f(x_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd}, \end{cases}$$

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so the sequence $\{f(x_n)\}$ diverges. ∴ By a contradiction of theorem 3.2.6. the limit does not exist. Great

3] If $D = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $f(x) = 2x+1 \;\forall\; x \in D$, evaluate $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ if possible. Explain your answer.

First I will do $\lim_{x\to 0} f(x)$. Notice that 0 is an accumulation Point of D.

Lemma 1

Let $\varepsilon>0$ be given. So choose an $x \in D \ni 0 < x < \varepsilon_1$. We can take the absolute value of x because since $0<x$ it will be positive. So we have $0<|x|<\varepsilon_1$. By additive identity we can say $0<|x-0|<\varepsilon_1$ which is the definition of accumulation point.

Now let $\varepsilon>0$ be given. Then take $\delta = \frac{\varepsilon}{2}$. So if $0<|x-0|<\delta = \frac{\varepsilon}{2}$ we can say that $|x-0|<\frac{\varepsilon}{2}$. Multiplying by 2 on both sides we get $2|x-0|<\varepsilon$. The two can be distributed inside the absolute value because it is positive, This gives $|2x-0|<\varepsilon$ we can replace 0 with $(1-1)$ to get $|2x-(1-1)|<\varepsilon$ or $|2x+1-1|<\varepsilon$. which is the same as $|f(x)-1|<\varepsilon$ as long as x is within δ of 1. Therefore the $\lim_{x\to 0} 2x+1$ is 1. $\square$

Next I will look at $\lim_{x\to 1} f(x)$, However I will show that 1 is not an accumulation point of D. therefore it does not make sense to talk about the limit as x approaches 1. I will show that the neighborhood of D_0 contains no other point of D other than D_0 itself.

Let $\varepsilon = \frac{1}{4}$ and since 1 is the point in question for accumulation an interval $(1-\frac{1}{4}, 1+\frac{1}{4}) = (\frac{3}{4}, \frac{5}{4})$ this contains only one point of D, namely 1 itself that is in the neighborhood hence there is no $x \in D \ni$ ~~$0<|x-D_0|$~~ $0<|x-1|<\varepsilon$ for any given $\varepsilon>0$

Beautiful!