1. Prove Theorem 4.1.7(a).

Suppose that $D$ is the domain of $f$.

(a) If $f$ is continuous at $a$, then there exists $\delta > 0$ such that $f$ is bounded on the set $(a-\delta, a+\delta) \cap D$.

Well, let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $|x-a| < \delta \implies |f(x) - f(a)| < \varepsilon$.

by Cor. 1.8.(b); $|f(x)| - |f(a)| \leq |f(x) - f(a)| < \varepsilon$

by transitive: $|f(x)| - |f(a)| < \varepsilon$

add $|f(a)|$: $|f(x)| < \varepsilon + |f(a)|$

Letting $M = \varepsilon + |f(a)|$ gives:

$|f(x)| < M$ for all $x \in (a-\delta, a+\delta) \cap D$.

Therefore $f(x)$ is bounded on the set $(a-\delta, a+\delta) \cap D$, as desired. □

Well done.
2) Th. 4.1.7 e

If \( D = (a, b) \), \( f \) is continuous at \( c \in D \), and \( f(c) > 0 \),
then there exists a neighborhood \( N_\varepsilon \) of \( c \) such that \( f(x) > 0 \)
for all \( x \in N_\varepsilon \cap (a, b) \).

Proof: Since \( f \) is continuous at \( c \), we know that for any \( \varepsilon > 0 \),
there exists a \( S > 0 \) such that \( |f(x) - f(c)| < \varepsilon \) for all \( |x - c| < S \)
and \( x \in (a, b) \). Since this is true for any \( \varepsilon > 0 \), and since \( f(c) > 0 \), we'll choose
\( \varepsilon \) such that \( 0 < \varepsilon < f(c) \). Now we have \( |f(x) - f(c)| < \varepsilon \),
which means by Theorem 1.8.5 that \( -\varepsilon < f(x) - f(c) < \varepsilon \), or
\( f(c) - \varepsilon < f(x) < f(c) + \varepsilon \). But since \( \varepsilon < f(c) \), \( f(c) - \varepsilon > 0 \).
Thus \( f(x) > f(c) - \varepsilon \) implies \( f(x) > 0 \) by transitivity.

All these statements hold for \( |x - c| < S \) and \( x \in (a, b) \), so
\(-S < x - c < S \), or \( c - S < x < c + S \). So for any \( 0 < \varepsilon < f(c) \),
there exists a neighborhood \( (c - S, c + S) \) such that \( f(x) > 0 \)
for \( x \in (c - S, c + S) \cap (a, b) \).

\( \square \)

Great Job!