1.) Prove Theorem 4.1.7(a).

Suppose that $D$ is the domain of $f$.

(a) If $f$ is continuous at $a$, then there exists $\delta > 0$ such that $f$ is bounded on the set $(a-\delta, a+\delta) \cap D$.

Well, let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$.

by Cor. 1.8.6(b): $|f(x)| - |f(a)| \leq |f(x)-f(a)| < \varepsilon$

by transitive: $|f(x)| - |f(a)| < \varepsilon$

add $|f(a)|$: $\quad |f(x)| < \varepsilon + |f(a)|$

Letting $M = \varepsilon + |f(a)|$ gives:

$\underline{|f(x)| < M}$ for all $x \in (a-\delta, a+\delta) \cap D$.

therefore $f(x)$ is bounded on the set $(a-\delta, a+\delta) \cap D$, as desired. $\square$

Well done.

2) Th. 4.1.7 e

If $D = (a, b)$, $f$ is continuous at $c \in D$, and $f(c) > 0$, then there exists a neighborhood $N_\varepsilon$ of $c$ such that $f(x) > 0$ for all $x \in N_\varepsilon \cap (a, b)$.

Proof: Since $f$ is continuous at $c$, we know that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $|x - c| < \delta$ and $x \in (a, b)$. Since this is true for <u>any</u> $\varepsilon > 0$, and since $f(c) > 0$, we'll choose $\varepsilon$ such that $0 < \varepsilon < f(c)$. Now we have $|f(x) - f(c)| < \varepsilon$, which means by Theorem 1.8.5 that $-\varepsilon < f(x) - f(c) < \varepsilon$, or $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. But since $\varepsilon < f(c)$, $f(c) - \varepsilon > 0$. Thus $f(x) > f(c) - \varepsilon$ implies $f(x) > 0$ by transitivity. All these statements hold for $|x - c| < \delta$ and $x \in (a, b)$, so $-\delta < x - c < \delta$, or $c - \delta < x < c + \delta$. So for any $0 < \varepsilon < f(c)$, there exists a neighborhood $(c - \delta, c + \delta)$ such that $f(x) > 0$ for $x \in (c - \delta, c + \delta) \cap (a, b)$. $\square$

Great Job!