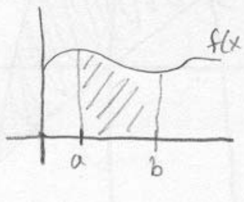
Exam 1 **Calc 2** **9/16/2005**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **State the formula** for volume by cylindrical shells when the region between f(x) and the x-axis from $x = a$ to $x = b$ is rotated around the y-axis.



$$\int_a^b 2\pi \, x \, f(x) \, dx$$

where x is the radius
and f(x) is the height

Yes!

2. **Set up an integral** for the average value of the function g(t) $= te^{-t^2}$ on the interval [0,5].

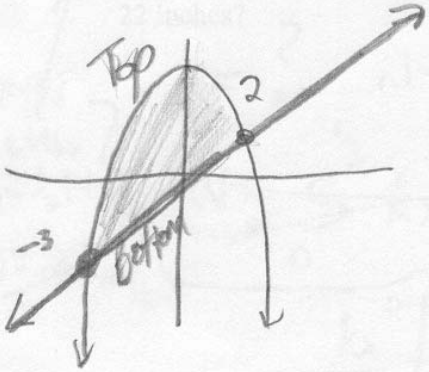
$$V_{avg} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

$$\frac{1}{5-0} \int_0^5 te^{-t^2}$$

Good

Ans: $\frac{1}{5} \int_0^5 te^{-t^2} dx$

3. **Find** the area of the region between $y = 6 - x^2$ and $y = x$.



$6 - x^2 = x$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or 2

$A = \int_{-3}^{2} T - B \, dx$

$A = \int_{-3}^{2} [(6 - x^2) - (x)] \, dx$

$A = (6 - x^2 - x) dx$

$A = 6x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-3}^{2}$

$= \left(6(2) - \frac{2^3}{3} - \frac{2^2}{2}\right) - \left(6(-3) - \frac{(-3)^3}{3} - \frac{(-3)^2}{2}\right)$

$= \left(12 - \frac{8}{3} - 2\right) - \left(-18 + 9 - \frac{9}{2}\right)$

$= \frac{22}{3} + \frac{27}{2}$

$= \frac{125}{6}$

Well done!

multiply by derivative of

4. If $F(x) = \int_0^x \sin^2 t \, dt$, what is $F'(x)$?

$F'(x) = \sin^2 x$ due to Fundamental Theory of Calculus

Good

5. A spring has a natural length of 16 inches, and 60 pounds hold it stretched to a length of 20 inches. How much work is done in stretching the spring from a length of 20 inches to a length of 22 inches?

$W = F \cdot h$

$f = kx$

$60\,lbs = k \cdot \frac{1}{3}\,ft$

$180 = k$

$\int_{\frac{1}{3}}^{.5} 180x\,dx = \left[90x^2\right]_{1/3}^{.5} =$

$\left[90(.5)^2\right] - \left[90(\tfrac{1}{3})^2\right]$

$22.5 \quad 10 \quad = 12.5$ ftlbs

Excellent

6. Evaluate $\int_0^{\pi/6} \sin\theta \cos^4\theta \, d\theta$.

let $u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$du = -\sin\theta \, d\theta$

$\frac{du}{-\sin\theta} = d\theta$

$\int_0^{\pi/6} \sin\theta \, u^4 \, d\theta$

$\cancel{\sin\theta}\, u^4 \cdot \frac{du}{-\cancel{\sin\theta}}$

$-u^4$

$-\frac{1}{5}u^5$

$\int_0^{\pi/6} -\frac{1}{5}\cos^5\theta$

$-\frac{1}{5}\cos^5\left(\frac{\pi}{6}\right) - \left(-\frac{1}{5}\cos^5(0)\right)$

Good Job

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. The professor gave us this huge *u*-substitution assignment, and she really emphasized that it's important, so I want to make sure to get it all right, you know? But so it's like 200 integrals, and half of them are the even-numbered problems, right? And she said it was really important that we check our answers, but my book doesn't have answers for the even ones. And so my friend asked the professor in class about it, 'cause I was afraid to, and the professor got really excited and kept saying that of course we could check our answers even if they're even. Is she just crazy?"

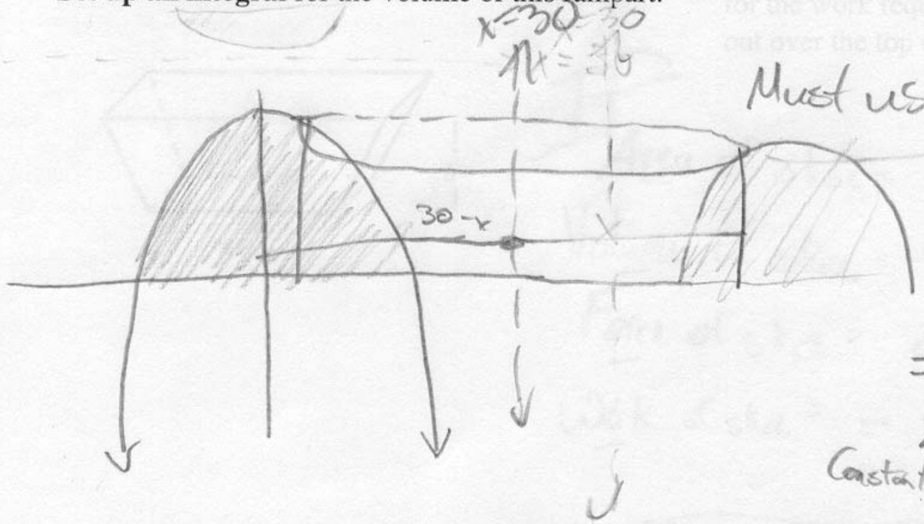
Explain clearly to Bunny how she might be expected to check her answers on such problems, even when the answers to the evens aren't in the back of the book.

Well Bunny I see you are in quite a quandry here but don't worry I can help you. When you integrate with u-substitution the easiest way to check your answers is not the back of the book but rather taking the derivative of your answer. When you integrate a function the derivative of your answer is the original function

If $\int f(x) = F(x)$, then $F'(x) = f(x)$

Well put!

8. Jon has decided to build a rampart around his house shaped like the solid formed by rotating the region under $f(x) = 9 - x^2$ but above the x axis around the axis $x = 30$ (with the scale in feet). **Set up an integral** for the volume of this rampart.



Must use the cylindrical shell method

$$V = \int_a^b 2\pi x f(x)\,dx$$

$$= 2\pi \int_a^b x f(x)\,dx$$

Constants Radius height

$$\text{Height} = (9 - x^2) - (0) = (9 - x^2)$$

$$\text{Radius} = 30 - x$$

$$9 - x^2 = 0$$
$$x^2 = 9$$
$$x = \pm 3$$

Points of intersection

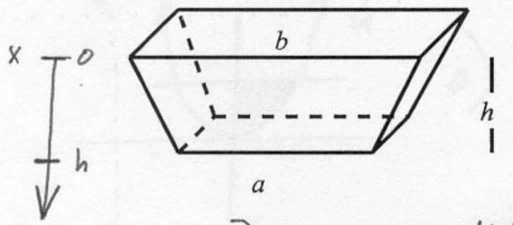
Therefore when plugged into equation

$$\boxed{V = 2\pi \int_{-3}^{3} \left[(30-x)(9^{2}-x^2)\right] dx}$$

in ft^3

Well done!

9. Suppose a pit shaped like an upside-down frustum of a pyramid is filled with water. If the pit has a square with side length b at the top and a square with side length a at the bottom, and a depth of h (with all of these measurements in meters, and water with a density of 1000kg/m^3), **set up an integral** for the work required to pump all the water out over the top edge of the pit.



x	y
0	b
h	a

First, find a linear relation giving the side length of a square horizontal slice.

$$m = \frac{a-b}{h}$$

$$y = \frac{a-b}{h}x + b$$

side length $= \frac{a-b}{h}x + b$ meters

area of a slice $= \left(\frac{a-b}{h}x + b\right)^2 \ m^2$

vol. of a slice $= \left(\frac{a-b}{h}x + b\right)^2 \Delta x \ m^3$

mass of a slice $= \left(\frac{a-b}{h}x + b\right)^2 \cdot \Delta x \cdot m^3 \cdot \frac{1000\text{kg}}{m^3}$

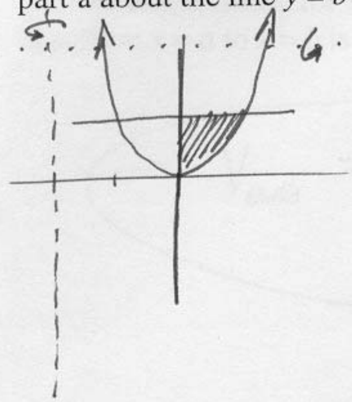
force for a slice $= \left(\frac{a-b}{h}x + b\right)^2 \cdot 1000\Delta x \cdot 9.8 \ N$

work for a slice $= 9800\left(\frac{a-b}{h}x + b\right)^2 x\Delta x \ J$

$$\text{Total work} = \int_0^h 9800x\left(\frac{a-b}{h}x + b\right)^2 dx$$

10. a) Let R be the first-quadrant region bounded by the y-axis, the curve $y = x^2$, and the line $y = 1$. Find the volume of the solid resulting when R is rotated about the line $x = -2$.

b) Find a positive value of b for which the volume obtained when rotating the region R from part a about the line $y = b$ is the same as that obtained in part a.



a) $$V_a = \int_0^1 2\pi(x+2)(1-x^2)\,dx$$

$$= 2\pi\int_0^1 (x - x^3 + 2 - 2x^2)\,dx$$

$$= 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4} + 2x - \frac{2}{3}x^3\right]_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4} + 2 - \frac{2}{3}\right) = \frac{19}{12}\cdot 2\pi = \frac{19\pi}{6}$$

b) $$V_b = \int_0^1 \pi(b-x^2)^2\,dx - \int_0^1 \pi(b-1)^2\,dx$$

$$= \int_0^1 \pi(b^2 - 2bx^2 + x^4)\,dx - \int_0^1 \pi(b^2 - 2b + 1)\,dx$$

$$= \pi\left[b^2x - \frac{2}{3}bx^3 + \frac{1}{5}x^5\right]_0^1 - \pi(b^2 - 2b + 1)x\Big|_0^1$$

$$= \pi\cdot b^2 - \pi\cdot\frac{2}{3}b + \frac{\pi}{5} - \pi b^2 + 2\pi b - \pi$$

$$= \frac{4}{3}\pi b - \frac{4}{5}\pi$$

So to make $V_a = V_b$:

$$\frac{4}{3}\pi b - \frac{4}{5}\pi = \frac{19\pi}{6}$$

$$\frac{4}{3}b = \frac{119}{30}$$

$$b = \frac{119}{40}$$