Exam 2 Calc 2 10/14/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int xe^x\,dx$. $= uv - \int v\,du$

$u = x \quad v = e^x$

$du = 1 \quad dv = e^x$

$= xe^x - \int e^x \cdot 1$

$\boxed{= xe^x - e^x + C}$

Good

2. **Set up** an integral for the length of the curve $y = \cos x$ on the interval $0 \le x \le 2\pi$.

$\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

$\cos' x = -\sin x$

$(-\sin x)^2 = \sin^2 x$

$\boxed{\int_0^{2\pi} \sqrt{1 + \sin^2 x}\,dx}$

Yes

3. Evaluate $\int \sin^3 x \cos^2 x\,dx$.

$\int \sin^2 x \sin x \cos^2 x\,dx$

$\int (1 - \cos^2 x)(\cos^2 x)(\sin x)\,dx$

$u = \cos x$

$dx = \frac{du}{-\sin x}$

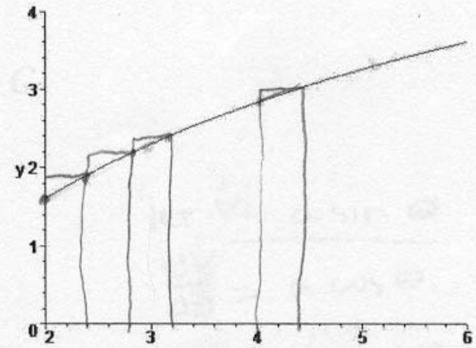
$\int (1 - u^2)(u^2)\cancel{(\sin x)}\frac{du}{\cancel{-\sin x}}$

$-\int u^2 - u^4\,du = -\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]$

$-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

Well done

4. When $\int_2^6 \ln\left(1+x^2\right)dx$ is approximated, $L_{10} \approx 10.637$, $R_{10} \approx 11.438$, and $M_{10} \approx 11.047$.



a) What is T_{10}, to three decimal places? $T = \frac{R+L}{2}$

11.038

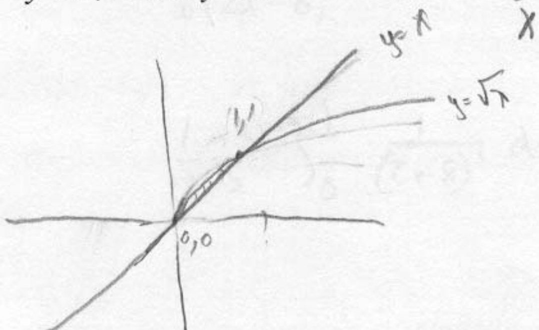
b) Which of these four approximations can you be certain is higher than the true area under the curve, and why?

We know that the right approx is bigger because when the rectangles are drawn there is extra space being counted that is not under the curve.

Excellent

We also know the midpoint approx is bigger because the function is concave down. When this happens you can draw a tangent line to the midpoint and the line will cover more space than the two rectangles do.

5. **Set up** integrals for the *x* coordinate of the center of mass of the region bounded by the curves $y=\sqrt{x}$ and $y=x$.



$\bar{x} = \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$ $a=0$ $b=1$

$y=\sqrt{x}$, $y=x$ $\Rightarrow$ $y=\sqrt{x}-x$ $= x^{1/2}-x^1$ $= x^{-1/2}$

$$\boxed{\bar{x} = \frac{\int_0^1 x\cdot(\sqrt{x}-x)\,dx}{\int_0^1 (\sqrt{x}-x)\,dx}}$$

Great

10

6. Derive the integration formula $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$.

$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

$$\boxed{\begin{aligned} &\text{let } u = a\sin\theta \\ &\frac{du}{d\theta} = a\cos\theta \\ &du = a\cos\theta\, d\theta \end{aligned}}$$

$\int \frac{du}{\sqrt{a^2-(a\sin\theta)^2}}$

Well done!

$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$

$\int \frac{a\cos\theta\, d\theta}{a\sqrt{(\cos^2\theta)}}$

$\int \frac{\cancel{a}\cancel{\cos\theta}\, d\theta}{\cancel{a}\cancel{\cos\theta}}$

$\int d\theta = \theta$

Now we must substitute back in for θ

$\frac{u}{a} = \frac{a\sin\theta}{a}$

$\frac{u}{a} = (\sin\theta)$ → we'll change to $\sin^{-1}$ to get the θ alone

$\theta = \boxed{\sin^{-1}\frac{u}{a} + C}$

7. Evaluate $\int_0^5 \frac{1}{(2x-8)^3}\,dx$.

Discontinuous where $x=4$!
Improper!

Break it up:

$$= \int_0^4 \frac{1}{(2x-8)^3}\,dx + \int_4^5 \frac{1}{(2x-8)^3}\,dx$$

$$= \lim_{b\to 4} \int_0^b \frac{1}{(2x-8)^3}\,dx +$$ Something else I'll only deal with if it turns out to matter...

let $u = 2x-8$

$\frac{du}{dx} = 2$

$\frac{du}{2} = dx$

$$= \lim_{b\to 4} \int_{x=0}^{x=b} u^{-3} \cdot \frac{du}{2}$$

$$= \lim_{b\to 4} \frac{1}{2} \cdot \frac{u^{-2}}{-2} \Big|_{x=0}^{x=b}$$

$$= \lim_{b\to 4} \frac{-1}{4} \cdot \frac{1}{(2x-8)^2} \Big|_0^b$$

$$= \lim_{b\to 4} \frac{-1}{4}\left[\frac{1}{(2b-8)^2} - \frac{1}{(-8)^2}\right]$$

$$= \lim_{b\to 4} \frac{-1}{4(2b-8)^2} + \frac{1}{256}$$

And as b approaches 4, that denominator approaches zero, so the fraction blows up!

Diverges

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dang, these trig integrals kick my butt. The sin and cos ones I was kinda getting, but then the tan and sec ones just blew me away. I was thinking it was about the same, like if you look for which one has an odd exponent or whatever, you know? But I tried the homework and that totally didn't work. How the heck do you even start those?"

Explain clearly to Biff how to approach integrals involving secant and tangent.

Look, Biff, if you're okay with the $\sin x$ and $\cos x$ integrals, you can learn to do the secant and tangent ones too. The only difference is because of the different derivatives. You know how with $\int \sin^3 x \cos^2 x\,dx$ you'd save a $\sin x$, because it would make that $u = \cos x$ substitution work? Well, so if you had to do one like $\int \tan^5 x \sec^2 x\,dx$, you can save the $\sec^2 x$ to make a $u = \tan x$ substitution work, since $\sec^2 x$ is the derivative of $\tan x$ (just like $\sin x$ is the derivative of $\cos x$ in the other example). This approach works any time the power on $\sec^n x$ is even, because you can use an identity to convert extra $\sec^2 x$ factors into tangents using the trig identity $\tan^2 x + 1 = \sec^2 x$ (which you can get from the $\sin^2 x + \cos^2 x = 1$ identity you already know, right?).

The other batch to know about are if the exponent on $\tan x$ is odd, you can save a $\sec x \tan x$ factor and you should be able to figure the rest out. Good Luck!

9. If $p(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.4e^{-0.4t} & \text{if } t \geq 0 \end{cases}$ is the probability density function for the number of minutes it takes Jon to find his pen after he starts looking for it,

a) Find the probability, to the nearest hundredth, that Jon finds his pen within 1 minute of beginning his search.

b) Find, to the nearest tenth of a minute, the median time it takes Jon to find his pen.

$u = .4t \quad \frac{du}{dt} = -.4 \quad dt = \frac{du}{-.4}$

a. $\int_0^1 .4e^{-.4t}\,dt = .4\int_0^1 e^{-.4t}\,dt$

$\int_0^1 .4e^u \frac{du}{-.4} = -\int_0^1 e^u\,du = -e^u\Big|_0^1 = -e^{.4t}\Big|_0^1 = -\left(\frac{1}{e^{.4}} - 1\right) = 1 - \frac{1}{e^{.4}} \approx 0.33$ or 33%

b. $\int_{-\infty}^{m} .4e^{.4t}\,dt = \frac{1}{2}$

Excellent!

$\lim_{b \to m} \int_0^b .4e^{-.4t} = 1/2$

$\lim_{b \to m} -e^{-.4t}\Big|_0^b = 1/2 \to -\left(\frac{1}{e^{.4b}} - 1\right) = \frac{1}{2} \to 1 - \frac{1}{e^{.4b}} = 1/2$

$\ln 1 - .4b = \ln 1/2$

median ≈ 1.7 minutes

10. Evaluate $\int \frac{2}{(x^2+1)(x-1)}\,dx$.

I wish: $\frac{2}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

$2 = (Ax+B)(x-1) + C(x^2+1)$

If $x=1$:

$2 = 0 + 2C$

$\therefore C = 1$

$2 = Ax^2 - Ax + Bx - B + 1x^2 + 1$

Matching x^2 coefficients:

$0 = A + 1$

$\therefore -1 = A$

Matching x coefficients:

$0 = -A + B$

$0 = -(-1) + B$

$-1 = B$

$= \int \left(\frac{-x-1}{x^2+1} + \frac{1}{x-1} \right) dx$

Split it up!

$= \int \left(\frac{-x}{x^2+1} + \frac{-1}{x^2+1} + \frac{1}{x-1} \right) dx$

$= -\frac{1}{2} \ln|x^2+1| - \tan^{-1} x + \ln|x-1| + C$

Through a little u-sub

'Cause you know it by heart

Through another little u-sub.