Exam 3 Calc 2 11/4/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formula for arc length in polar coordinates.

Arc Length = $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

2. Determine whether $y = 2 + 5e^{2x+3}$ is a solution to the differential equation $\frac{dy}{dx} = 2y - 4$.

$y' = 10e^{2x+3}$

sub for y

$10e^{2x+3} \stackrel{?}{=} 2(2+5e^{2x+3}) - 4$

$10e^{2x+3} \stackrel{?}{=} 4 + 10e^{2x+3} - 4$

$10e^{2x+3} = 10e^{2x+3}$

Great

Yes, $y = 2+5e^{2x+3}$ is a solution.

3. Find the exact coordinates of all points on the curve with parametric equations $x(t) = t^3 - 12t$, $y(t) = 10 - t^2$ where the tangent line is vertical.

Tangent line is vertical where $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = x'(t) = 3t^2 - 12$$

$$\frac{dx}{dt} = 3(t^2 - 4)$$

$$t^2 - 4 = 0 \quad t^2 = 4$$

$$t = \pm 2$$

Great

$$x(2) = 8 - 24 = -16 \qquad x(-2) = -8 + 24 = 16$$

$$y(2) = 6 \qquad y(-2) = 6$$

$$\boxed{\begin{matrix}(-16, 6)\\(16, 6)\end{matrix}}$$

4. Find a general solution to the differential equation $\dfrac{dH}{dt} = k(212 - H)$.

$$\frac{dH}{dt} = k(212 - H)$$

$$\int \frac{1}{(212 - H)}\, dH = \int k\, dt$$

$$-\ln(212 - H) = kt + C$$

$$\ln(212 - H) = (-kt + C)$$

$$212 - H = e^{(-kt + C)}$$

$$-H = (-212 + Ae^{-kt})$$

$$H = (212 - Ae^{-kt})$$

Well done

5. Classify the graph with equation $4x^2 + 9y^2 + 108 = 72y$ as a parabola, ellipse, or hyperbola, and sketch the graph, labeling the exact coordinates of at least four points.

$$4x^2 + 9y^2 + 108 = 72y$$

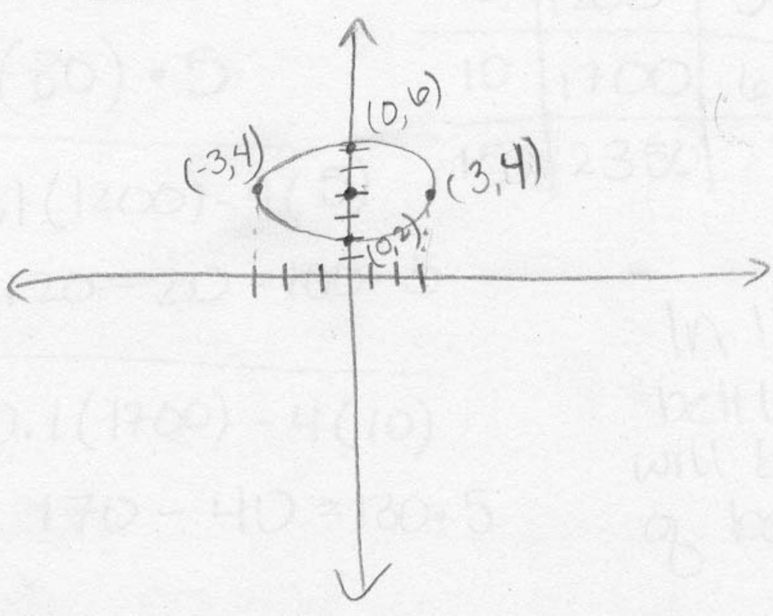
$$4x^2 + 9y^2 - 72y = -108$$

$$4x^2 + 9(y^2 - 8y + 16) = -108 + 144$$

$$\frac{4x^2 + 9(y-4)^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{9} + \frac{(y-4)^2}{4} = 1 \therefore \text{an ellipse}$$

Excellent



6. A company is trying to market an insecticide to combat the invasion of Asian beetles in Iowa, claiming that it will kill the beetles at an increasing rate every year until they are eliminated. Suppose that the situation is modeled by the differential equation $\frac{dP}{dt} = 0.1P - 4t$. Use Euler's method with $\Delta t = 5$ to approximate the Asian beetle population 15 years from the present, provided that the present population is 800 (with all the population numbers actually measured in millions of beetles).

$$\frac{dP}{dt} = 0.1P - 4t \qquad \Delta t = 5$$

$P(0) = 800$ millions of betles

$$\frac{\Delta P}{\Delta t} = 0.1P - 4t$$

$$\frac{\Delta P}{\Delta t} = .1(800) - 4(0)$$

$$\Delta P = (80) \cdot 5$$

$$\frac{\Delta P}{\Delta t} = 0.1(1200) - 4(5)$$

$$120 - 20 = 100 \cdot 5$$

$$\frac{\Delta P}{\Delta t} = 0.1(1700) - 4(10)$$

$$\Delta P = 170 - 40 = 130 \cdot 5$$

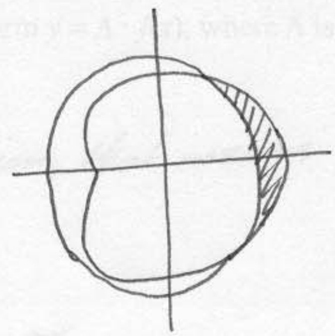
Δt	P	ΔP
0	800	400
5	1200	500
10	1700	650
15	2350	

Great Job

In 15yrs the Asian bettle population will be 2,350 millions of bettles

8. Find the area of the region inside the graph with polar equation $r = 2 + 4\cos\,\theta$ but outside the graph with polar equation $r = 4$.

Where do they intersect?



$$4 = 2 + 4\cos\theta$$

$$2 = 4\cos\theta$$

$$\frac{1}{2} = \cos\theta$$

$$\theta = \pm\frac{\pi}{3}$$

$$\text{So Area} = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(2 + 4\cos\theta\right)^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4)^2 d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4 + 16\cos\theta + 16\cos^2\theta\right) d\theta - \frac{1}{2}\cdot 16\theta \Big|_{-\pi/3}^{\pi/3}$$

$$= \frac{4}{4}\int_{-\pi/3}^{\pi/3} \left(2 + 8\cos\theta + 8\cos^2\theta\right) d\theta - 8\cdot\frac{\pi}{3} + 8\cdot\frac{-\pi}{3}$$

$$= \Big[2\theta + 8\sin\theta + 8\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)\Big]_{-\pi/3}^{\pi/3} - \frac{16\pi}{3}$$

$$= \left[2\left(\frac{\pi}{3}\right) + 8\sin\left(\frac{\pi}{3}\right) + 4\left(\frac{\pi}{3}\right) + 2\sin 2\left(\frac{\pi}{3}\right)\right]$$

$$- \left[2\left(\frac{-\pi}{3}\right) + 8\sin\left(\frac{-\pi}{3}\right) + 4\left(\frac{-\pi}{3}\right) + 2\sin 2\left(\frac{-\pi}{3}\right)\right] - \frac{16\pi}{3}$$

$$= \frac{6\pi}{3} + 8\cdot\frac{\sqrt{3}}{2} + 2\,\frac{\sqrt{3}}{2} + 6\,\frac{\pi}{3} + 8\,\frac{\sqrt{3}}{2} + 2\,\frac{\sqrt{3}}{2} - \frac{16\pi}{3}$$

$$= 4\pi + 10\sqrt{3} - \frac{16\pi}{3} = 10\sqrt{3} - \frac{4\pi}{3}$$

9. Suppose that you know a function $y = f(x)$ is a solution to the differential equation

$$a y' + b y = 0$$

where a and b are constants. Show that any other function of the form $y = A \cdot f(x)$, where A is a constant, will also be a solution to that differential equation.

if $y = f(x)$ is solution of that function

$\Rightarrow y' = f'(x)$

$\Rightarrow a \cdot f'(x) + b f(x) = 0.$

And if $y = A f(x)$.

$\Rightarrow y' = A f'(x)$ because A = constant

$\Rightarrow a \cdot y' + by = 0.$

$\Leftrightarrow a \, A f'(x) + b \cdot A f(x) = 0.$

$= A \cdot (a f'(x) + b f(x))$

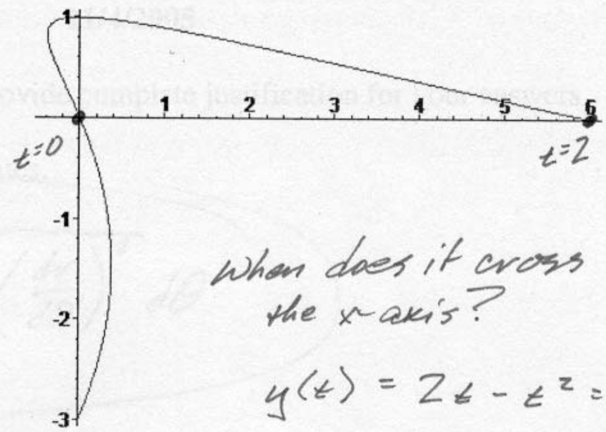
But $a f'(x) + b f(x) = 0$

$\Rightarrow A \cdot 0 = 0$

Excellent

$\Rightarrow y = A f(x)$ also is solution of the equation

10. Find the area of the region bounded by the graph of the parametric functions $x(t) = t^3 - t$, $y(t) = 2t - t^2$ and the x-axis.



$$\text{Area} = \int_{\alpha}^{\beta} y(t) \cdot x'(t)\,dt$$

$$= \int_0^2 (2t - t^2)\cdot(3t^2 - 1)\,dt$$

$$= \int_0^2 (6t^3 - 2t - 3t^4 + t^2)\,dt$$

$$= \left[\frac{6}{4}t^4 - t^2 - \frac{3}{5}t^5 + \frac{1}{3}t^3\right]_0^2$$

$$= 24 - 4 - \frac{96}{5} + \frac{8}{3}$$

$$= \frac{52}{15}$$

When does it cross the x-axis?

$y(t) = 2t - t^2 = 0$

$t(2-t) = 0$

$t = 0$ and $t = 2$