Exam 4 Calc 2 12/2/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the first 3 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

$a_1 = \frac{1}{3}$

$a_2 = \frac{1}{3^2} = \frac{1}{9}$ Good

$a_3 = \frac{1}{3^3} = \frac{1}{27}$

$S_1 = \frac{1}{3}$

$S_2 = \frac{1}{3} + \frac{1}{9}$

$S_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$

2. Determine whether the sequence $\left\{\frac{3+5n^2}{n+n^2}\right\}$ converges or diverges, and if it converges find its limit.

$$\lim_{n\to\infty} \frac{5n^2+3}{n^2+n} \overset{L'H}{=} \lim_{n\to\infty} \frac{10n}{2n+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{10}{2} = 5$$

Conver

To test to see if sequence converges take the limit of sequence to find out if the terms are approaching any number + what number

Excellent

$\left\{\frac{3+5n^2}{n+n^2}\right\}$ converges to 5

3. Determine the sum of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. $\left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \right\}$

geometric series where $a = \frac{1}{3}$ and $r = \frac{1}{3}$

$S = \frac{a}{1-r}$ when $|r| < 1$ $\frac{1}{3}$ is less than 1

$$S = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Excellent

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ converges or diverges.

Use <u>Integral test</u>

$f(x) = \frac{x^2}{x^3+1}$ so $\lim_{b \to \infty} \int_1^b \frac{x^2}{x^3+1}\, dx$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$\lim_{b \to \infty} \int_1^b \frac{\cancel{x^2}}{1+u} \frac{du}{3\cancel{x^2}} = \frac{1}{3} \ln(1+u) \Big|_1^b$

$\lim_{b \to \infty} \frac{1}{3} \ln(1+u) \Big|_1^b = \infty - \ln 2 =$ <u>diverges</u>

By the <u>Integral Test</u> $\lim_{b \to \infty} \int_1^b \frac{x^2}{x^3+1}$ <u>diverges</u> so the

$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ must also <u>diverge</u> Nice!

5. Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges or diverges.

use <u>A.S.T!</u>

✓ (1) $\frac{1}{n \ln n}$ is <u>decreasing</u>? derivative: $\frac{0 \cdot n \ln n - (\ln n + 1)}{(n \ln n)^2} = \frac{-\ln n - 1}{(n \ln n)^2}$

This is negative for all positive n because the <u>negative</u> ln on <u>top</u> is always negative + the <u>bottom</u> (a square) is always <u>positive</u>.

✓ (2) $\lim_{n \to \infty} b_n = 0$? $\lim_{n \to \infty} \frac{1}{n \ln(n)} = 0$ we know this because the constant on top and <u>ever-increasing</u> number on <u>bottom</u> (both ln(n) and (ln(n))) are increasing for positive n)

Therefore, by A.S.T. the series <u>converges</u>

Excellent

6. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n \cdot 2^n}$ converges or diverges.

Use Rat. Test

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)(2^{n+1})}}{\frac{1}{n \cdot 2^n}} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)(2^{n+1})} \cdot \frac{n \cdot 2^n}{1} \right| = \frac{n}{2(n+1)}$$

$$\lim_{n\to\infty} \left| \frac{n}{2(n+1)} \right| = \lim_{n\to\infty} \frac{n}{2n+2} \overset{L'H}{=} \frac{1}{2}$$

so the $\lim_{n\to\infty}$ of $\sum_{n=0}^{\infty} \frac{1}{n \cdot 2^n} = \frac{1}{2}$ and by the Ratio Test

we know that if the lim is < 1 the series

converges absolutely, so therefore $\sum_{n=0}^{\infty} \frac{1}{n \cdot 2^n}$

converges absolutely.

Nice

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, this is so crazy. Pretty much the only thing I've figured out so far about this series crap is that the series one over n is supposed to diverge, right? So we had this quiz and it had $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ on it, and I saw the one over n in there and knew it had to diverge, right? So I said it compared to the one over n thing so it had to diverge, but the T.A. wrote a bunch of crap I didn't understand, and I got no credit at all. What's up with that?"

Explain clearly to Biff what's wrong with his version of "comparison".

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \dots$$

Here are the rules of the comparison test Biff:

If $a_n \le b_n$ & $\sum b_n$ converges, then $\sum a_n$ also converges

If $a_n \le b_n$ & $\sum a_n$ diverges, then $\sum b_n$ is also divergent

In your case Biff, let $a_n = \sum_{n=1}^{\infty} \frac{1}{n}$, which you know diverges, and let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

When you compare terms in the serieses you notice that all terms in b_n are smaller than the ones in a_n.

In other words $b_n \le a_n$. In the rules of the comp. test, it has to be $a_n \le b_n$, which you don't have here.

Therefore knowing that a_n or $\sum \frac{1}{n}$ diverges doesn't tell you anything when b_n or $\sum \frac{1}{n^2+n}$ is less than a_n.

Excellent

8. Determine the interval of convergence for the power series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$.

Rat. Test

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\dfrac{(-1)^n x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+1}}\, x^{\cancel{2n+2}}}{(2n+2)\cancel{(2n)!}\cdot(2n+1)}\cdot\frac{\cancel{2n!}}{\cancel{(-1)^n}\,\cancel{x^{2n}}}\right| =$$

$4n^2+\cancel{2n+4n}+2$

← constant $6n$

$$\lim_{n\to\infty}\left|\frac{-x}{4n^2+6n+2}\right| \overset{L'H}{=} \frac{0}{8n+6} = 0$$

Excellent

$(-\infty,$

interval of convergence is $(-\infty,\infty)$ because any x-value works

9. Use a 9$^{\text{th}}$ degree polynomial to approximate $\int_0^1 \sqrt{1-x^3}\,dx$. [Hint: Starting by finding a series for $f(x)=\sqrt{1+x}$ might be a good plan].

$f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2}$ $\quad f''(0) = \frac{-1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\quad f'''(0) = \frac{3}{8}$

So $\sqrt{1+x} \approx 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2 + \frac{3/8}{6}x^3$, and substituting $(-x^3)$

gives $\sqrt{1+(-x^3)} \approx 1 - \frac{x^3}{2} - \frac{x^6}{8} - \frac{x^9}{16}$.

So $\int_0^1 \sqrt{1-x^3}\,dx \approx \int_0^1 \left(1 - \frac{x^3}{2} - \frac{x^6}{8} - \frac{x^9}{16}\right)dx$

$$= \left[x - \frac{x^4}{8} - \frac{x^7}{56} - \frac{x^{10}}{160}\right]_0^1$$

$$= 1 - \frac{1}{8} - \frac{1}{56} - \frac{1}{160}$$

10. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series, will $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ converge? Why or why not?

Nope. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ both converge by the A. S. T.,

but $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}} \cdot \frac{(-1)^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}\cdot\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n}$

is the harmonic series, which diverges.