Exam 4   Calc 2    12/2/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the first 3 partial sums of the series \( \sum_{n=1}^{\infty} \frac{1}{3^n} \).
   \[
   a_1 = \frac{1}{3} \\
   a_2 = \frac{1}{3^2} = \frac{1}{9} \\
   a_3 = \frac{1}{3^3} = \frac{1}{27}
   \]
   \[
   S_1 = \frac{1}{3} \\
   S_2 = \frac{1}{3} + \frac{1}{9} \\
   S_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{12}{27} \]

2. Determine whether the sequence \( \frac{3+5n^2}{n+n^2} \) converges or diverges, and if it converges find its limit.
   \[
   \lim_{n \to \infty} \frac{3+5n^2}{n+n^2} = \lim_{n \to \infty} \frac{10n}{2n+1} = \lim_{n \to \infty} \frac{10}{2} = 5
   \]

   To test to see if sequence converges take the limit of sequence to find out if the terms are approaching any number + what number

   Excellent

   \[ \sum_{n=1}^{\infty} \frac{3+5n^2}{n+n^2} \] converges to 5

3. Determine the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{3^n} \).
   \[
   \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{3}
   \]

   geometric series where \( a = \frac{1}{3} \) and \( r = \frac{1}{3} \)

   \[
   S = \frac{a}{1-r} \text{ when } |r| < 1 \quad \frac{1}{3} \text{ is less than 1}
   \]

   \[
   S = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}
   \]

   Excellent
4. Determine whether the series \( \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \) converges or diverges.

Use **Integral test**

\[
 f(x) = \frac{x^2}{x^3+1} \quad \text{so} \quad \lim_{b \to \infty} \int_1^b \frac{x^2}{x^3+1} \, dx \\
\lim_{b \to \infty} \int_1^b \frac{x^2}{x^3+1} \, dx = \frac{1}{3} \left( \lim_{b \to \infty} (1 + u) \right) = \infty
\]

By the Integral Test, \( \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \) diverges so the series \( \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \) must also diverge. **Nice!**

5. Determine whether the series \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n} \) converges or diverges.

Use **A.S.T.**

\( \sqrt{1} \) \( \frac{1}{n \ln n} \) is decreasing? derivative: \( \frac{0 - \ln n - \ln n + 1}{(n \ln n)^2} = \frac{-\ln n + 1}{(n \ln n)^2} \)

This is negative for all positive \( n \) because the negative \( \ln \) on top is always negative and the bottom (a square) is always positive.

\( \sqrt{2} \) \( \lim_{n \to \infty} b_n = 0? \)

\( \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \) we know this because the constant on top and ever-increasing number on bottom (both \( n \) and \( \ln(n) \)) are increasing for positive \( n \).

Therefore, by **A.S.T.** the series converges.

**Excellent**
6. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n \cdot 2^n}$ converges or diverges.

Use Ratio Test

$$\left| \lim_{n \to \infty} \frac{\frac{1}{n+1} \cdot 2^{n+1}}{\frac{1}{n} \cdot 2^n} \right| = \left| \lim_{n \to \infty} \frac{1}{(n+1)(2^{n+1})} \cdot \frac{n \cdot 2^n}{1} \right| = \frac{n}{2(n+1)}$$

$$\left| \lim_{n \to \infty} \frac{n}{2(n+1)} \right| = \lim_{n \to \infty} \frac{n}{2n+2} = \frac{1}{2}$$

So the limit $\lim_{n \to \infty} \frac{1}{n \cdot 2^n} = \frac{1}{2}$ and by the Ratio Test we know that if the limit is less than 1 the series converges absolutely, so therefore $\sum_{n=0}^{\infty} \frac{1}{n \cdot 2^n}$ converges absolutely.
7. Biff is a calculus student at Enormous State University, and he’s having some trouble. Biff says “Dude, this is so crazy. Pretty much the only thing I’ve figured out so far about this series crap is that the series one over n is supposed to diverge, right? So we had this quiz and it had
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + n}
\]
on it, and I saw the one over n in there and knew it had to diverge, right? So I said it compared to the one over n thing so it had to diverge, but the T.A. wrote a bunch of crap I didn’t understand, and I got no credit at all. What’s up with that?”

**Explain clearly** to Biff what’s wrong with his version of “comparison”.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots \right)
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots
\]

Here are the rules of the comparison test Biff:

- If \( a_n \leq b_n \leq c_n \) converges, then \( c_n \) also converges.
- If \( a_n \leq b_n \leq c_n \) diverges, then \( c_n \) is also divergent.

In your case Biff, let \( a_n = \frac{\pi}{n} \), which you knew diverges, and let \( b_n = \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \).

When you compare terms in the series, you notice that all terms in \( b_n \) are smaller than the ones in \( a_n \).

In other words \( b_n \leq a_n \). In the rules of the comparison test, it has to be \( a_n \leq b_n \), which you don’t have here.

Therefore knowing that \( a_n \) or \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges doesn’t tell you anything when \( b_n \) or \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \) is less than \( a_n \).

**Excellent**
8. Determine the interval of convergence for the power series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \).

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)(2n+1)x^{2n+2}} \right| = \frac{2n}{4n^2 + 2n + 2} = \frac{2n}{4n^2 + 2n + 2} = \frac{2n}{4(n^2 + 1)} \to 0.
\]

Hence, the interval of convergence is \((-\infty, \infty)\) because any \(x\)-value works.
9. Use a 9th degree polynomial to approximate \( \int_{0}^{\infty} \sqrt{1-x^3} \, dx \). [Hint: Starting by finding a series for \( f(x) = \sqrt{1+x} \) might be a good plan].

\[
\begin{align*}
  f(x) &= \sqrt{1+x} \\
  f'(x) &= \frac{1}{2} (1+x)^{-\frac{1}{2}} \\
  f''(x) &= -\frac{1}{4} (1+x)^{-\frac{3}{2}} \\
  f'''(x) &= \frac{3}{8} (1+x)^{-\frac{5}{2}}
\end{align*}
\]

\[
\begin{align*}
  \sqrt{1+x} &\approx 1 + \frac{1}{2} x + \frac{1}{2} x^2 + \frac{3}{8} x^3, \quad \text{and substituting } (x^3) \\
  \text{gives } \sqrt{1+(x^3)} &\approx 1 - \frac{x^3}{2} - \frac{x^6}{8} - \frac{x^9}{16}
\end{align*}
\]

\[
\begin{align*}
  \int_{0}^{\infty} \sqrt{1-x^3} \, dx &\approx \int_{0}^{\infty} \left(1 - \frac{x^3}{2} - \frac{x^6}{8} - \frac{x^9}{16}\right) \, dx \\
  &= \left[ x - \frac{x^4}{8} - \frac{x^7}{56} - \frac{x^{10}}{160} \right]_0^\infty \\
  &= 1 - \frac{1}{2} - \frac{1}{56} - \frac{1}{160}
\end{align*}
\]

10. If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are both convergent series, will \( \sum_{n=1}^{\infty} (a_n \cdot b_n) \) converge? Why or why not?

Nope. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) and \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) both converge by the A.S.T., but \( \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n}} \cdot \frac{(-1)^n}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \)

is the harmonic series, which diverges.