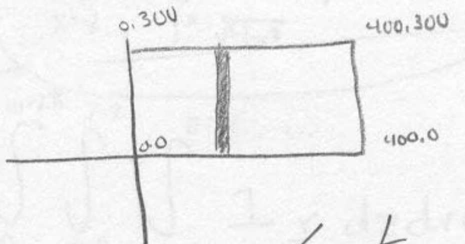
Exam 2 Calc 3 10/28/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Jon plans to do some research about the elk population in the state of Colorado. He estimates that if you regard Colorado as a rectangle, with its lower left corner at the origin, lower right corner at the point (400, 0), upper left corner at the point (0, 300), and upper right corner at the point (400, 300), then the moose population density as of October 1st, 2005, was roughly given by the function $v(x, y) = 6 + .01x - 0.02y$ elk per square mile. Write an iterated integral for the total elk population in Wyoming.

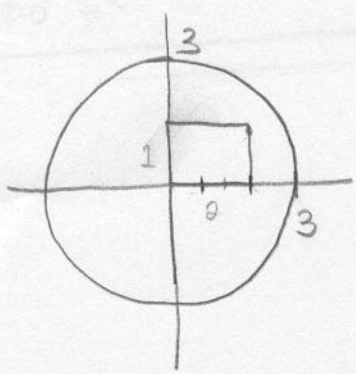


$$\int_{x=0}^{x=400} \int_{y=0}^{y=300} (6 + .01x - .02y)\,dy\,dx$$

Great

2. Set up an iterated integral for the volume of the region beneath the surface $z = 9 - x^2 - y^2$ and above the rectangle in the xy-plane with vertices at the origin, (2,0), (2,1), and (0,1).

$z = 9 - x^2 - y^2$ $\quad x^2 + y^2 = 9 \Leftarrow$ circle radius 3



$$\boxed{\int_{x=0}^{x=2} \int_{y=0}^{y=1} \int_{0}^{9-x^2-y^2} 1\,dz\,dy\,dx}$$

Great

Cone

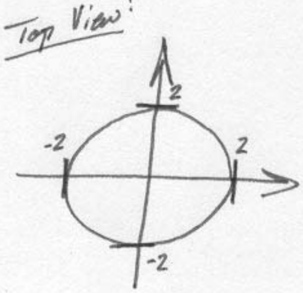
Cylinder around z axis with radius 2

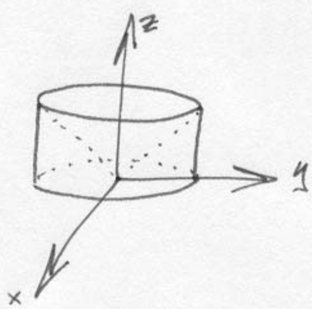
3. Set up an iterated integral for the region bounded by the xy-plane, the surface $x^2 + y^2 = 4$, and the surface $z = \sqrt{x^2 + y^2}$ in at least **two** of the following coordinate systems:

(a) Rectangular (b) Cylindrical (c) Spherical

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{x^2+y^2}} 1\, dz\,dy\,dx \qquad \int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{r} 1\cdot r\, dz\,dr\,d\theta \qquad \int_{0}^{2\pi}\int_{\pi/4}^{\pi/2}\int_{0}^{2/\sin\phi} 1\cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Top View:

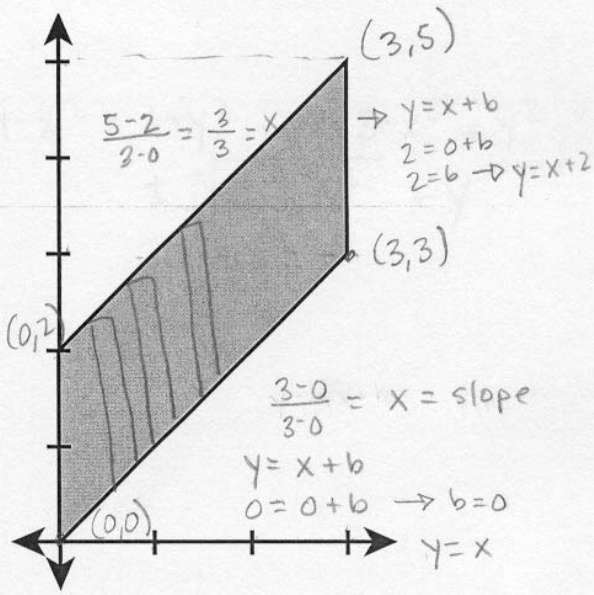




4. Suppose that $\delta(x,y) = 3 - 0.2x + 0.5y$ gives the density in grams per square cm at a point (x,y) of a strip of pink metal shaped like the region shown at right. Set up an integral for the total mass of the strip.

$$\int_{x=0}^{x=3}\int_{y=x}^{y=x+2} (3 - 0.2x + 0.5y)\,dy\,dx$$

Great



5. Set up an iterated integral **in cylindrical coordinates** for the volume of the region bounded by the hyperboloid of two sheets $z^2 - x^2 - y^2 = 1$ and the plane $z = 2$.

$z^2 = 1 + x^2 + y^2 = 1 + r^2$

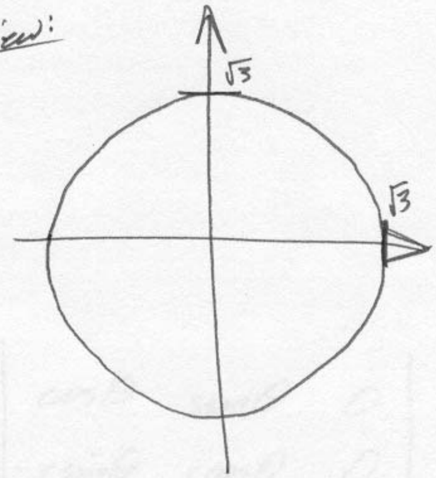
Where do they intersect?

$(2)^2 - x^2 - y^2 = 1$

$3 = x^2 + y^2$

is a circle with radius $\sqrt{3}$!

Top View:



$$\text{Volume} = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\sqrt{3}}\int_{z=\sqrt{1+r^2}}^{z=2} 1 \; r \, dz\, dr\, d\theta$$

6. Compute the Jacobian for the transformation to <u>cylindrical coordinates</u>.

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x \quad y \quad z$

$r \quad \theta \quad z$

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

Using trig identity $\cos^2\theta + \sin^2\theta = 1$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

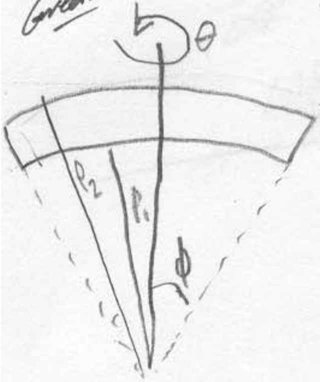
Excellent

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, I'm so totally lost in calculus. Our T.A. was talking about, like, he did this summer research project, and they did, like, the amount of ozone over the South Pole, you know, 'cause there's a hole or something. So he was saying they set up integrals for it and everything, but that's gotta be messed up, 'cause even with these triple integrals it's still only really good for stuff that's like rectangles, right? And the atmosphere is definitely *not* a rectangle, even I know that."

Explain clearly to Biff how spherical coordinates (he saw them in a previous chapter, but doesn't yet know what they have to do with integration) could be used in a situation like this.

Spherical coordinate are actually really easy for this. You'd first have to figure out how thick the atmosphere actually is and then you'd use limits for the bottom of the atmosphere to the top that will be a rho (ρ) value

Great!



and then figure out over what area of atmosphere which needs a phi (ϕ) value

And then 0 to 2π for a complete revolution.

And it looks like this

$$\int_0^{2\pi} \int_0^{\Delta\phi} \int_{\rho_1}^{\rho_2} f \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

8. Set up an iterated integral for the volume of the region bounded below by the surface $z = x^2$ and above by the surface $z = a - y^2$, where a is some positive constant.

$x^2 + y^2 = a$

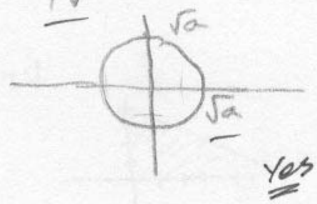
$y = \sqrt{a - x^2}$

$x^2 = a - y^2$

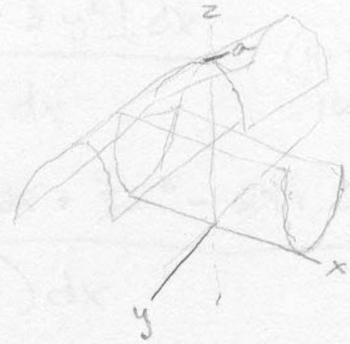
$$\int_{-\sqrt{a}}^{\sqrt{a}} \int_{-\sqrt{a-x^2}}^{\sqrt{a-x^2}} \int_{x^2}^{a-y^2} 1 \; dz\, dy\, dx$$

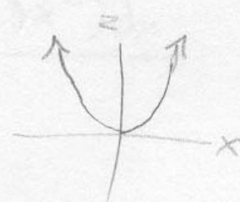
Excellent

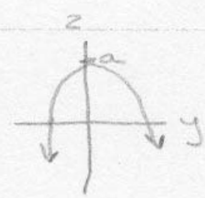
TV



Yes





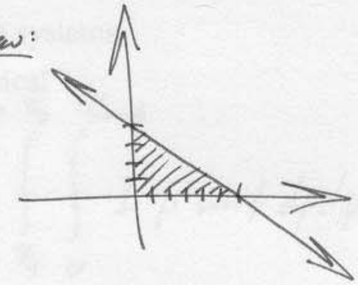


9. Find the z coordinate of the center of mass of the tetrahedron formed by the first-octant region below the surface $2x + 3y + z = 12$.

Where does it hit the xy-plane?

$$2x+3y+(0)=12$$
$$3y=12-2x$$
$$y=\frac{-2}{3}x+4$$

Top View:



$$\text{So}\quad \bar{z} = \frac{\int_0^6\int_0^{4-\frac{2}{3}x}\int_0^{12-2x-3y} z\,dz\,dy\,dx}{\int_0^6\int_0^{4-\frac{2}{3}x}\int_0^{12-2x-3y} 1\,dz\,dy\,dx}$$

We know from class that this volume is $\frac{1}{3}b\cdot h$, or $\frac{1}{3}\left(\frac{1}{2}\cdot 6\cdot 4\right)\cdot 12 = 48$

$$=\frac{1}{48}\cdot\int_0^6\int_0^{4-\frac{2}{3}x}\left.\frac{z^2}{2}\right|_0^{12-2x-3y}dy\,dx$$

let $u=12-2x-3y$

$\frac{du}{dy}=-3$

$dy=\frac{du}{-3}$

$$=\frac{1}{96}\int_0^6\int_{y=0}^{y=4-\frac{2}{3}x}u^2\cdot\frac{du}{-3}\,dx$$

$$=\frac{1}{96}\cdot\frac{-1}{3}\int_0^6\left.\frac{u^3}{3}\right|_{y=0}^{y=4-\frac{2}{3}x}dx$$

$$=\frac{-1}{864}\int_0^6\left\{\left[12-2x-3\left(4-\tfrac{2}{3}x\right)\right]^3-\left[12-2x-3(0)\right]^3\right\}dx$$

$$=\frac{-1}{864}\int_0^6\left[(0)^3-(12-2x)^3\right]dx$$

let $u=12-2x$

$\frac{du}{dx}=-2$

$dx=\frac{du}{-2}$

$$=\frac{-1}{864}\int_{x=0}^{x=6}-u^3\cdot\frac{du}{-2}$$

$$=\frac{-1}{864}\cdot\frac{+1}{2}\cdot\left.\frac{u^4}{4}\right|_{x=0}^{x=6}$$

$$=\frac{-1}{1728}\cdot\frac{1}{4}\cdot\left[(12-2x)^4\right]_0^6$$

$$=\frac{-1}{6912}\left[(0)^4-(12)^4\right]$$

$$=\frac{12^4}{6912}=3$$

10. The integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$ is extremely important in probability and many applications of multivariable integration. It's also quite troublesome. Evaluate this integral by converting it to polar coordinates and working it in that form.

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} \cdot r\,dr\,d\theta$$

let $u = -r^2$

$\frac{du}{dr} = -2r$

$\frac{du}{-2r} = dr$

$$= \int_0^{2\pi} \lim_{b\to\infty} \int_{r=0}^{r=b} e^{u} \cdot \cancel{r} \cdot \frac{du}{-2\cancel{r}}\,d\theta$$

$$= \int_0^{2\pi} \lim_{b\to\infty} \cdot \frac{-1}{2}\, e^{u} \Big|_{r=0}^{r=b}\,d\theta$$

$$= \int_0^{2\pi} \lim_{b\to\infty} \frac{-1}{2} \cdot \left(e^{-b^2} - e^{0}\right) d\theta$$

$$= \frac{-1}{2}\int_0^{2\pi} (0-1)\,d\theta$$

$$= \frac{-1}{2} \cdot -\theta \Big|_0^{2\pi}$$

$$= \frac{1}{2} \cdot 2\pi - \frac{1}{2} \cdot 0$$

$$= \pi$$