Exam 1 Calc 2 9/15/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Show how an appropriate trig substitution can transform the integral $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ into the integral $\int 1 d\theta$.

$\int \frac{1}{\sqrt{a^2-x^2}} = \int \frac{1}{\sqrt{a^2-(a\sin\theta)^2}} \cdot a\cos\theta\, d\theta$

when using trig substitution and you have a^2-x^2 you use $a\sin\theta$, so

$= \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}} d\theta = \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}} d\theta$

$x = a\sin\theta$

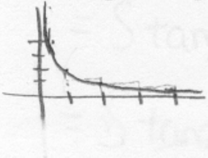
$dx = a\cos\theta\, d\theta$

$\int \frac{a\cos\theta}{\sqrt{a^2}\sqrt{\cos^2\theta}} d\theta = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \boxed{\int 1\, d\theta}$

Nice!

2. Use midpoint approximations with $n = 3$ subdivisions to approximate $\int_1^4 \frac{1}{x}\,dx$.

$n = \frac{4-1}{3}$



$$\int_1^4 \frac{1}{x}\,dx$$

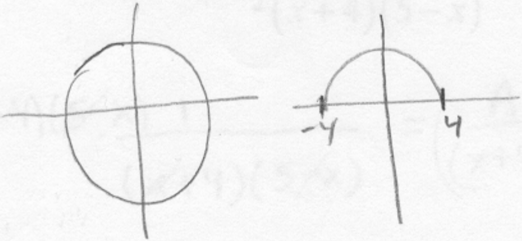
$$= \frac{4-1}{3}\left(\frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5}\right)$$

$$= 1\left(\frac{2}{3} + \frac{2}{5} + \frac{2}{7}\right)$$

$$= \boxed{1.35}$$

Great

3. **Set up** an integral for the circumference of a circle with radius 4.



$$\boxed{2\int_{-4}^{4}\sqrt{1+\left(\frac{-x}{\sqrt{4^2-x^2}}\right)^2}\ dx}$$

$x^2+y^2=4^2$

$\sqrt{4^2-x^2}=\sqrt{y^2}$

$\sqrt{4^2-x^2}=y=(4^2-x^2)^{1/2}$

Arc length $= \int_a^b \sqrt{1+(f'(x))^2}$

$y'=\frac{1}{2}(4^2-x^2)^{-1/2}\cdot(-2x)$

$y'=\frac{-x}{\sqrt{4^2-x^2}}$

Great

4. Evaluate $\int \tan^3 x \sec x\, dx$.

$$\int \tan^3 x \sec x\, dx = \int \tan^2 x (\tan x) \sec x\, dx$$

$$= \int (\sec^2 x - 1) \tan x \sec x\, dx$$

$$= \int (u^2-1) \cancel{\tan x \sec x} \cdot \frac{du}{\cancel{\sec x \tan x}}$$

$$= \int (u^2-1)\, du$$

$$= \frac{u^3}{3} - u + C$$

plug back in the value for u.

$$= \boxed{\tfrac{1}{3}\sec^3 x - \sec x + C}$$

$$\frac{\sin^2 x + \cos^2 x = 1}{\cos^2 x}$$

$$= \tan^2 x + 1 = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

Let $u = \sec x$. Then

$$\frac{du}{dx} = \sec x \tan x,$$

$$dx = \frac{du}{\sec x \tan x}$$

Well done!

5. Evaluate $\int \frac{1}{(x+4)(5-x)}\,dx$.

$$\frac{1}{(x+4)(5-x)} = \frac{A}{x+4} + \frac{B}{5-x}$$

$$1 = A(5-x) + B(x+4)$$

Excellent

If $x=5$ $\quad B = 1/9$

If $x=-4$ $\quad A = 1/9$

$$\int \frac{\frac{1}{9}}{x+4}\,dx + \int \frac{\frac{1}{9}}{5-x}\,dx$$

$$\int \frac{\frac{1}{9}}{x+4}\,dx \qquad \text{let } u = x+4 \qquad \frac{du}{dx} = 1 \qquad + \quad \frac{1}{9}\int \frac{1}{5-x}\,dx \qquad \text{let } u = 5-x \qquad \frac{dy}{dx} = -1$$

$$= \frac{1}{9}\int \frac{1}{u}\,du \qquad - \frac{1}{9}\int \frac{1}{u}\,du$$

$$= \boxed{\frac{1}{9}\ln|x+4| - \frac{1}{9}\ln|5-x| + C}$$

Mean $= \int_a^b x\,p(x)\,dx$

6. Let $p(x) = \begin{cases} 12x^2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \le 0 \text{ or } x \ge 1 \end{cases}$. Find the mean of this probability density function.

$$\underbrace{\lim_{a \to -\infty} \int_{-a}^{0} x \cdot 0\,dx}_{0} + \underline{\int_0^1 x(12x^2(1-x))\,dx} + \underbrace{\lim_{b \to \infty} \int_1^b x \cdot 0\,dx}_{0}$$

$$= \int_0^1 (12x^3 - 12x^4)\,dx$$

$$= \frac{12x^4}{4} - \frac{12x^5}{5} \Big]_0^1$$

$$= 3x^4 - \frac{12}{5}x^5 \Big|_0^1$$

$$= 3 - \frac{12}{5} = \frac{3}{5}$$

Excellent

7. Find the x coordinate of the center of mass of the first-quadrant portion of a circle centered at the origin with radius 3.

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x\cdot f(x)\,dx}$$

And for a circle with radius 3,

$A = \pi(3)^2 = 9\pi$,

so the first-quadrant portion has area $\frac{9\pi}{4}$

$$\bar{x} = \frac{1}{9\pi/4}\int_0^3 x\sqrt{9-x^2}\,dx$$

let $u = 9-x^2$

$\frac{du}{dx} = -2x$

$\frac{du}{-2x} = dx$

$$= \frac{4}{9\pi}\int_{x=0}^{x=3} x\cdot u^{1/2}\cdot\frac{du}{-2x}$$

$$= \frac{-2}{9\pi}\int_{x=0}^{x=3} u^{1/2}\,du$$

$$= \frac{-2}{9\pi}\cdot\frac{2}{3}\,u^{3/2}\Big|_{x=0}^{x=3}$$

$$= \frac{-4}{27\pi}\left(9-x^2\right)^{3/2}\Big|_0^3$$

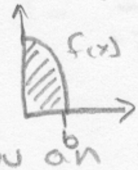
$$= \frac{-4}{27\pi}\cdot 0 - \frac{-4}{27\pi}\cdot 27$$

$$= \frac{4}{\pi}$$

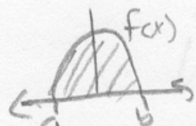
8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "This calculus stuff is totally messed up. It's like, sometimes they do these integrals by they say it's symmetry and they just double part of it or whatever, right? But I did that on this one question on our test, where it was like the center of mass of a parabola, and got totally the wrong answer, and since it's multiple choice I don't even get partial credit. But I totally don't get why I was wrong, 'cause the right half and left half are exactly alike, right?"

Explain clearly to Biff what's wrong with what he did, and under what circumstances using symmetry might be suspect.

When you found the $\bar{x}$ value for the right half of the parabola you found the x value for the center of mass of this shape:

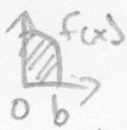


when you doubled ~~#~~ your answer it gave you an x-value even farther to the right. If you look at the graph



it is clear the center of mass should be somewhere at $x=0$. In order to get this you must use the entire area as the limits of integration. It always helps to know what your integral is actually calculating.

If Biff was calculating the area under the curve his integral of $\int_0^b f(x)$ would give all the area under the right portion of the parabola →



It is in this scenario that he could use its symetrical properties to double his answer and get the entire area.

Don't be fooled by ~~wat~~ what the integral is actually calculating.

Excellent!

9. Derive the formula $\int \sin^{-1} u\, du = u \sin^{-1} u + \sqrt{1-u^2} + C$ from the table of integrals.

let $x = \sin^{-1} u$ $\qquad$ $v = u$

$dx = \dfrac{1}{\sqrt{1-u^2}}$ $\qquad$ $dv = 1$

$= u \sin^{-1} u - \displaystyle\int \frac{u}{\sqrt{1-u^2}}\, du$ $\qquad$ let $x = 1-u^2$

$dx = -2u\, du$

$-\frac{1}{2} dx = u\, du$

$= u \sin^{-1} u - \frac{1}{2} \displaystyle\int \frac{1}{\sqrt{x}}\, du$ $\qquad$ $x^{-1/2} \to 2x^{1/2}$

$= u \sin^{-1} u - \left(-\frac{1}{2} \cdot 2x^{1/2} \right) + c$

$= u \sin^{-1} u + (1-u^2)^{1/2} + c$

Well done.

$\boxed{= u \sin^{-1} u + \sqrt{1-u^2} + c}$

10. Find the surface area resulting when the curve $y = e^{-x}$ to the right of $x = 0$ is rotated around the x-axis.

$$\boxed{\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)}\,dx}$$

$y = e^{-x}$

$y' = -e^{-x}$

let $u = e^{-x}$

$\frac{du}{dx} = -e^{-x}$

$-e^{x}\,du = dx$

$$= \int_0^\infty 2\pi (e^{-x})\sqrt{1 + (-e^{-x})^2}\,dx$$

$$= \lim_{b\to\infty} 2\pi \int_0^b e^{-x}\sqrt{1 + e^{-2x}}\,dx$$

$$= \lim_{b\to\infty} 2\pi \int_{x=0}^{x=b} \cancel{e^{-x}}\sqrt{1 + u^2}\cdot -\cancel{e^{x}}\,du$$

$$= \lim_{b\to\infty} -2\pi \int_{x=0}^{x=b} \sqrt{1+u^2}\,du$$

By table line 21

$$= \lim_{b\to\infty} -2\pi \left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_{x=0}^{x=b}$$

$$= \lim_{b\to\infty} -2\pi \left[\frac{e^{-x}}{2}\sqrt{1+e^{-2x}} + \frac{1}{2}\ln\left(e^{-x} + \sqrt{1+e^{-2x}}\right)\right]_0^b$$

$$= \lim_{b\to\infty} -2\pi \left\{\left[\frac{e^{-b}}{2}\sqrt{1+e^{-2b}} + \frac{1}{2}\ln\left(e^{-b} + \sqrt{1+e^{-2b}}\right)\right] - \left[\frac{e^{0}}{2}\sqrt{1+e^{0}} + \frac{1}{2}\ln\left(e^{0} + \sqrt{1+e^{0}}\right)\right]\right\}$$

$$= \lim_{b\to\infty} -\pi \left[\frac{\sqrt{1+e^{-2b}}}{e^{b}} + \ln\left(e^{-b} + \sqrt{1+e^{-2b}}\right) - \sqrt{2} - \ln\left(1+\sqrt{2}\right)\right]$$

Using L'Hôpital on the first term

$$= \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$