Exam 3 **Calc 2** **11/3/2006**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Convert the polar point $(5, \pi/2)$ to rectangular coordinates.

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

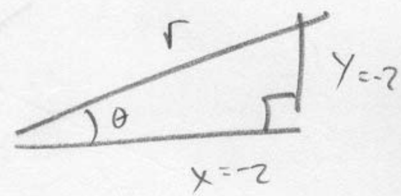
$x = 5\cos\frac{\pi}{2}$ $\quad$ $y = 5\sin\frac{\pi}{2}$

$x = 5(0) = 0$ $\quad$ $y = 5(1) = 5$

$\underline{(0, 5)}$

b) Convert the rectangular point $(-2, -2)$ to polar coordinates.

Great



$r^2 = (-2)^2 + (-2)^2 =$

$r^2 = 4 + 4$

$r = \pm\sqrt{8}$

$\tan\theta = \frac{-2}{-2} = 1$

$\theta = \frac{\pi}{4}$ in III quad

$\left(\sqrt{8}, \frac{5\pi}{4}\right)$ or $\left(-\sqrt{8}, \frac{\pi}{4}\right)$

2. Determine whether $y = x - x^{-1}$ is a solution to the differential equation $x y' + y = 2x$.

$y = x - \frac{1}{x}$ $\quad$ $\underline{y' = \frac{1}{x^2} + 1}$

so $\quad x\left(\frac{1}{x^2} + 1\right) + \left(x - \frac{1}{x}\right) \overset{?}{=} 2x$

$\frac{x}{x^2} + x + x - \frac{1}{x} \overset{?}{=} 2x$

$\cancel{\frac{1}{x}} + x + x - \cancel{\frac{1}{x}} \overset{?}{=} 2x$

Great

$\underline{2x = 2x}$

$\underline{\text{yes it is a solution}}$

3. Find an equation for the line tangent to the curve with parametric equations

$$x = t^4 + 1$$
$$y = t^3 + t$$

at the point corresponding to $t = -1$.

$x' = 4t^3$

$y' = 3t^2 + 1$

$\frac{dy}{dx} = \frac{3t^2+1}{4t^3}$

$\frac{dy}{dx} = \frac{3(-1)^2+1}{4(-1)^3} = \frac{4}{-4} = -1$

slope of tangent line when $t=-1$

$x = (-1)^4 + 1$ $\quad$ $y = (-1)^3 + (-1)$

$x = 2$ $\quad$ $y = -2$

$y + 2 = -1(x-2)$

$y = -x + 2$

Well done

4. Use Euler's method with a step size of $\Delta x = 0.2$ to approximate $y(0.4)$ for the differential equation $y' = 1 - xy$ if $y(0) = 0$.

$\frac{dy}{dt} = 1 - 0 \cdot 0 \Rightarrow dy = 0.2$

$\frac{dy}{dt} = 1 - (0.2)(0.2) \Rightarrow dy = 0.192$

$1 - 0.04$

0.96

Δx	y	Δy
0	0	0.2
0.2	0.2	0.192
0.4	0.392	

$\therefore\ y(0.4) = 0.392$

Good

$(y-2)^2$

$y^2 - 4y + 4$

5. Find the vertices of the conic section $9x^2 + 25y^2 - 100y = 125$.

$9x^2 + 25(y^2 - 4y + 4) = 125 + 100$

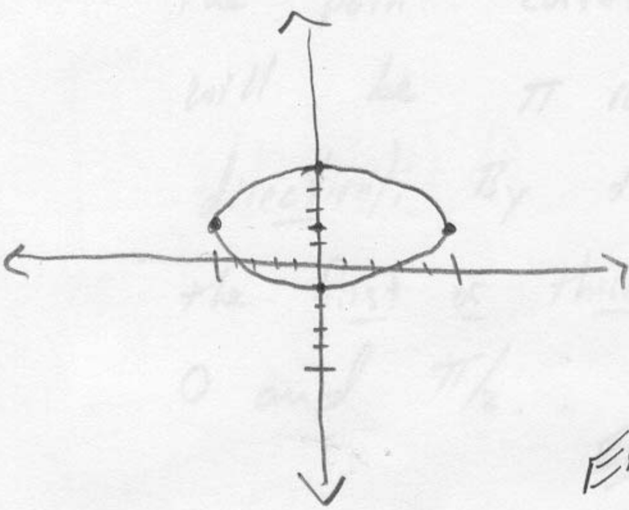
$\left(9x^2 + 25(y-2)^2 = 225\right) \frac{1}{225}$

$\frac{x^2}{25} + \frac{(y-2)^2}{9} = 1$

$x = \pm 5$

$y = \pm 3$

center @ $(0,2)$



vertices @

$(0,5)$, $(0,-1)$

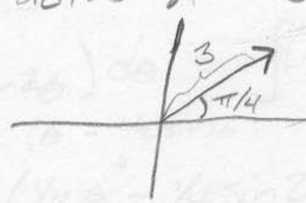
$(5,2)$, $(-5,2)$

Excellent

6. Bunny is a calculus student at Enormous State University, and she's still having some trouble. Bunny says "Ohmygod, this is so totally confusing! We learned about polar coordinates, and I thought I understood it, you know? But then our exam was all true/false, 'cause they have a machine do it, you know? So there was this question if any point in polar with the theta between zero and pi over two has to be in the first quadrant, you know? So, like, I said true, because that's where angles like that are, right? But it came back marked wrong, and the professor said not to come ask questions about the test because he'll lower your score. So what's up with *that*?

Explain clearly to Bunny what the possibilities are for polar points with theta values between zero and pi over two.

Bunny, you're correct, on a normal graph of x, y and your unit circle, any thing from zero to $\pi/2$ **IS** in the first quadrant. The thing about polar cordinates is, you can go both forwards **and** **backwards** once you found an angle to start at. Let me show you. Let's take a radius of 3 and $\pi/4$. This could give you



. That's what you'd normally think, but we can have a -3 go with a $\pi/4$ giving you

. See how this works? My $\pi/4$ was in the right quadrants I just didn't travel forwards.

Excellent

7. Set up an integral for the area inside the inner loop of the curve with polar equation $r = 1 + 2\sin\theta$.

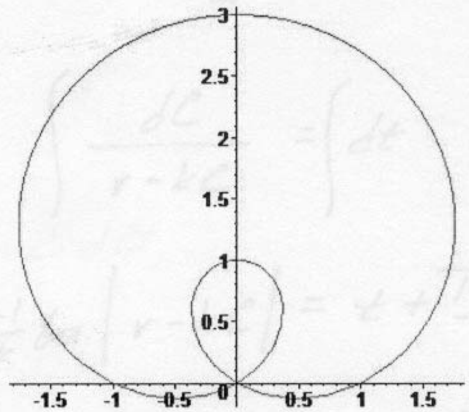
Where does it cross zero?

$0 = 1 + 2\sin\theta$

$-1 = 2\sin\theta$

$\frac{-1}{2} = \sin\theta$

$\frac{-\pi}{6} = \theta$



So $\frac{11}{6}\pi$ and $\frac{7}{6}\pi$ bound the inner loop, and

$$\text{Area} = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(1+2\sin\theta)^2\, d\theta$$

8. The concentration of some chemical in the bloodstream is given by $\frac{dC}{dt} = r - kC$. Find a general solution to this differential equation.

$$\frac{dC}{dt} = r - kC$$

$$dC = (r - kC)\,dt$$

$r - kC = u$

$-k = \frac{du}{dC}$

$-\frac{1}{k}du = dC$

$$\int \frac{1}{r-kC}\,dC = \int dt$$

$$-\frac{1}{k}\ln|r-kC| = t + D$$

$-k \cdot D = E$

$$\ln|r-kC| = -kt + E$$

$e \qquad e$

$$|r-kC| = e^{-kt} \cdot e^{E}$$

$\pm e^{E} = A$

$$r - kC = Ae^{-kt}$$

$$-kC = Ae^{-kt} - r$$

$$\boxed{C = \frac{-A}{k}e^{-kt} + \frac{r}{k}}$$

Excellent

9. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, to find the area that it encloses.

here
Area

$$A = \int_0^{2\pi} y(\theta)\cdot x'(\theta)\cdot d\theta$$

$$= \int_0^{2\pi} b\sin\theta \cdot -a\sin\theta \cdot d\theta$$

$$= -ab\int_0^{2\pi} \sin^2\theta \cdot d\theta$$

$$= -ab\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\frac{ab}{2}\int_0^{2\pi} (1-\cos 2\theta)\, d\theta$$

$$= -ab/2 \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -ab/2 \left[2\pi - \left(\frac{\sin 4\pi}{2} - \frac{\sin 0}{2}\right)\right]$$

$$= -ab/2 \cdot 2\pi = -\pi ab$$

[The negative sign is because the direction of movement is in negative direction]

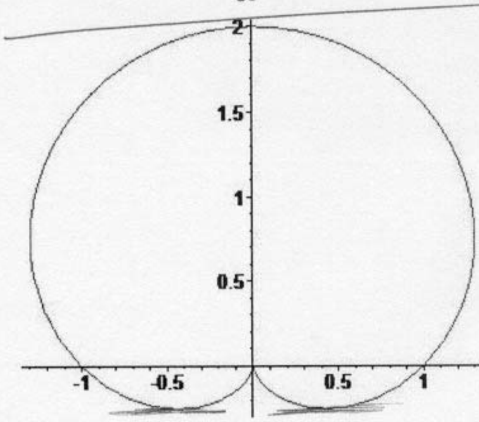
$\therefore$ Actual area $= \pi ab$

Well done

10. Find the rectangular coordinates of the lowest point(s) on the graph with polar equation

$$r = 1 + \sin\theta.$$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r'\cdot\sin\theta + r\cos\theta}{\text{something I don't need}}$



The lowest points will be places where the tangent line is horizontal, so the numerator of $\frac{dy}{dx}$ is zero, so

$0 = (\cos\theta)(\sin\theta) + (1+\sin\theta)(\cos\theta)$

$0 = \sin\theta\cos\theta + \cos\theta + \sin\theta\cos\theta$

$0 = \cos\theta\,(1 + 2\sin\theta)$

So either $\cos\theta = 0$ or $1 + 2\sin\theta = 0$

$\theta = \frac{\pi}{2}$ or $2\sin\theta = -1$

$\sin\theta = -1/2$

$\theta = -\pi/6$ And from symmetry, $\theta = \frac{7\pi}{6}$ is the other.

So that point is:

$r = 1 + \sin\left(\frac{-\pi}{6}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{-\pi}{6}\right)$

or:

$r = 1 + \sin\left(\frac{7\pi}{6}\right) = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{7\pi}{6}\right)$

Or in rectangular:

$x = \left(\frac{1}{2}\right)\cos\left(\frac{-\pi}{6}\right) = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

$y = \left(\frac{1}{2}\right)\sin\left(\frac{-\pi}{6}\right) = \frac{1}{2}\cdot\frac{-1}{2} = \frac{-1}{4}$

$x = \left(\frac{1}{2}\right)\cos\left(\frac{7\pi}{6}\right) = \frac{1}{2}\cdot\frac{-\sqrt{3}}{2} = \frac{-\sqrt{3}}{4}$

$y = \left(\frac{1}{2}\right)\sin\left(\frac{7\pi}{6}\right) = \frac{1}{2}\cdot\frac{-1}{2} = \frac{-1}{4}$

$\left(\frac{\sqrt{3}}{4}, \frac{-1}{4}\right)$ and $\left(\frac{-\sqrt{3}}{4}, \frac{-1}{4}\right)$