Exam 1 Calc 3 10/6/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(*x*, *y*) with respect to *y*.

The partial derivative of $f_y(x,y)$:

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Good

2. Find an equation for the plane tangent to $z = x^2 + y^2$ at the point (3,–2).

$z - z_0 = f_x(x,y)(x-x_0) + f_y(x,y)(y-y_0)$

$f_x = 2x \qquad f_x(3,-2) = 2(3) = 6$

$f_y = 2y \qquad f_y(3,-2) = 2(-2) = -4$

$z = 3^2 + (-2)^2$

$z = 9 + 4$

$z_0 = 13$

$z - 13 = 6(x-3) - 4(y+2)$

$z = 6x - 18 - 4y - 8 + 13$

$z = 6x - 4y - 13$

Great

3. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Approach along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

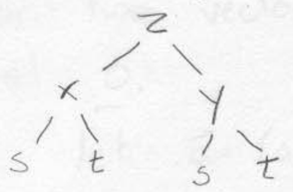
Approach along $x=y$

$$\lim_{(y,y)\to(0,0)} \frac{y\cdot y}{y^2+y^2} = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}$$

Great

Since along different approaches we approach different heights the limit does not exist.

4. Suppose $z = f(x,y)$, where $x = g(s,t)$, $y = h(s,t)$, $g(1,2) = 3$, $g_s(1,2) = -1$, $g_t(1,2) = 4$, $h(1,2) = 6$, $h_s(1,2) = -5$, $h_t(1,2) = 10$, $f_x(3,6) = 7$, and $f_y(3,6) = 8$. Find $\partial z/\partial t$ when $s = 1$ and $t = 2$.



$$\frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}$$

$s=1 \qquad t=2$

$$\frac{dx}{dt} = g_t(s,t) \qquad \frac{dz}{dx} = f_x(g(s,t), h(s,t))$$

$$\frac{dy}{dt} = h_t(s,t) \qquad \frac{dz}{dy} = f_y(g(s,t), h(s,t))$$

$$\text{so } \frac{dz}{dt} = g_t(s,t)\cdot f_x(g(s,t),h(s,t)) + h_t(s,t)\cdot f_y(g(s,t),h(s,t))$$

plug in values

$$g_t(1,2)\cdot f_x(3,6) + h_t(1,2)\cdot f_y(3,6) \qquad s=1 \quad t=2$$

$$\frac{dz}{dt} = (4)\cdot(7) + (10)\cdot(8) = 28 + 80 = \underline{108}$$

Good

5. Let $f(x,y) = y^2/x$. Find the maximum rate of change of f at the point (2,4) and the direction in which it occurs.

Max rate of change @ pt (2,4) = magnetude of gradient

direction = gradient

$$f_x = y^2 \cdot 1x^{-2} = -\frac{y^2}{x^2} = -\frac{4^2}{2^2} = -\frac{16}{4} = -4$$

$$f_y = \frac{2y}{x} = \frac{2\cdot 4}{2} = \frac{8}{2} = 4$$

$$\text{gradient} = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle = \langle -4, 4 \rangle$$ - direction

$$|\text{grad}| = \sqrt{(-4)^2 + 4^2} = \sqrt{16+16} = \sqrt{32}$$ = max rate of change

Excellent!

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

$$\text{So, } \vec{a} \times \vec{b} \cdot \vec{a} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} + \cancel{a_2 a_3 b_1} - \cancel{a_1 a_2 b_3} + \cancel{a_1 a_3 b_2} - \cancel{a_2 a_3 b_1}$$

$$= 0$$

The dot product of any two vectors that are $\perp = 0$, $\therefore$ $\vec{a} \times \vec{b}$ is $\perp$ to $\bar{a}$.

Nice!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. The professor is such a total geek. It's like he keeps saying we're supposed to know what this stuff means instead of just finding the right answers, you know? But everybody knows math isn't like that. But just in case, I guess I should kinda have some clue, you know? So like directional derivatives are one of the things where he said we should know what it means, and I'm totally stumped. I mean, it's just a formula, right?"

   Explain clearly to Bunny what directional derivatives mean.

Directional derivatives are the rate of change (the slope, if you will) on a 3-dimensional surface in any direction you choose. We know the slope in the x direction is f_x and the slope in the y direction is f_y, but how about halfway between?

We can express this "diagonal" direction as a vector, $\langle a, b\rangle$. If we're halfway between x and y axes, we could say $\langle 1, 1\rangle$. It makes sense that the slope in that direction would be how far you're going in the x-direction times the x-slope, plus how far you're going in the y-direction times the y-slope. So, we have $f_x a + f_y b$. However, we need to make $\langle a, b\rangle$ a unit vector (magnitude 1) or else $f_x a + f_y b$ could be made arbitrarily large, even if we're facing the same direction! (for instance, $\langle 1,1\rangle$ and $\langle 5000, 5000\rangle$ face in the same direction.) So, we make $\langle a, b\rangle$ ~~at~~ a unit vector: $\langle 1,1\rangle \to \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$. Also, note that $f_x a + f_y b$ can also be written $\langle f_x, f_y\rangle \cdot \langle a, b\rangle$ by definition of dot product.

Great answer.

8. Find the maximum and minimum values of the function $f(x, y) = 3x - 2y$ subject to the constraint $x^2 + y^2 = 5$.

$g(x,y)$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\langle 3, -2 \rangle \quad \langle 3, -2 \rangle = \lambda \langle 2x + 2y \rangle$

$x^2 + y^2 = 5 \quad 3 = \lambda 2x \quad \lambda = \frac{3}{2x}$

$-2 = \lambda 2y \quad \lambda = -\frac{1}{y}$

$\frac{3}{2x} = \frac{-1}{y} \quad 3y = -2x \quad y = -\frac{2}{3}x \qquad x = \frac{3}{2}y$

$x^2 + (-\frac{2}{3}x)^2 = 5$

$x^2 + \frac{4}{9}x^2 = 5 \qquad \frac{13}{9}x^2 = 5 \qquad x = \pm\sqrt{\frac{45}{13}}$

$\left(\sqrt{\frac{45}{13}}\right)^2 + y^2 = 5 \qquad \frac{45}{13} + y^2 = \frac{65}{13} \qquad y = \pm\sqrt{\frac{20}{13}}$

$f\left(\sqrt{\frac{45}{13}}, \sqrt{\frac{20}{13}}\right) = 3\sqrt{\frac{45}{13}} - 2\sqrt{\frac{20}{13}}$ (+) + (-) = smaller than max, larger than min

$f\left(-\sqrt{\frac{45}{13}}, -\sqrt{\frac{20}{13}}\right) = -3\sqrt{\frac{45}{13}} + 2\sqrt{\frac{20}{13}}$ (-) + (+) = larger than minium, smaller than max

$f\left(\sqrt{\frac{45}{13}}, -\sqrt{\frac{20}{13}}\right) = 3\sqrt{\frac{45}{13}} + 2\sqrt{\frac{20}{13}}$ (+) + (+) = maximum

$f\left(-\sqrt{\frac{45}{13}}, \sqrt{\frac{20}{13}}\right) = -3\sqrt{\frac{45}{13}} - 2\sqrt{\frac{20}{13}}$ (-) - (+) = minimum

$\boxed{\text{maximum } \left(\sqrt{\frac{45}{13}}, -\sqrt{\frac{20}{13}}\right), \text{ minimum } \left(-\sqrt{\frac{45}{13}}, \sqrt{\frac{20}{13}}\right)}$

Great.

9. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ at the point (x_0, y_0, z_0) can be written as $\dfrac{x\,x_0}{a^2}+\dfrac{y\,y_0}{b^2}+\dfrac{z\,z_0}{c^2}=1$.

The slickest way:

let $f(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

so $\nabla f(x,y,z) = \left\langle \dfrac{2x}{a^2}, \dfrac{2y}{b^2}, \dfrac{2z}{c^2} \right\rangle$

and $\nabla f(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$.

Then this vector is normal to the function's level surface $f(x,y,z)=1$, so the plane through (x_0,y_0,z_0) with this normal vector is what we need:

$$\frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + \frac{2z_0}{c^2}(z-z_0) = 0$$

Now this can be rewritten:

$$\frac{2x_0x}{a^2} - \frac{2x_0^2}{a^2} + \frac{2y_0y}{b^2} - \frac{2y_0^2}{b^2} + \frac{2z_0z}{c^2} - \frac{2z_0^2}{c^2} = 0$$

Dividing through by 2 is easy. Realizing that (x_0, y_0, z_0) was a point on the ellipsoid and thus $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2} = 1$ is less easy, but gives:

$$\frac{x\,x_0}{a^2} + \frac{y\,y_0}{b^2} + \frac{z\,z_0}{c^2} = 1.$$

10. Determine the coordinates of the vertex of the paraboloid $z = x^2 + y^2 + axy + bx + cy + d$, where a, b, c, and d all represent constant real values. Is this vertex a maximum or minimum value, and how do you know?

I. Find partials:

$$z_x = 2x + ay + b \qquad z_{xx} = 2$$

$$z_{xy} = a$$

$$z_y = 2y + ax + c \qquad z_{yy} = 2$$

II. Set 'em equal 0:

$$0 = 2x + ay + b \Rightarrow x = \frac{-ay-b}{2}$$

$$0 = 2y + ax + c$$

$$0 = 2y + a\left(\frac{-ay-b}{2}\right) + c$$

$$0 = 2y + \frac{-a^2}{2}y + \frac{-ab}{2} + c$$

$$\frac{ab}{2} - c = y\left(2 - \frac{a^2}{2}\right)$$

$$y = \frac{\frac{ab}{2} - c}{2 - \frac{a^2}{2}} = \frac{ab - 2c}{4 - a^2}$$

$$\text{so} \quad x = \frac{-a\left(\frac{ab-2c}{4-a^2}\right) - b}{2}$$

$$= \frac{-a^2b + 2ac}{4-a^2} \cdot \frac{1}{2} - \frac{b}{2}\frac{(4-a^2)}{(4-a^2)}$$

$$= \frac{2ac - \cancel{a^2b} - 4b + \cancel{a^2b}}{8 - 2a^2}$$

$$= \frac{ac - 2b}{4 - a^2}$$

III. Second derivative test:

$$D(x,y) = f_{xx} \cdot f_{yy} - f_{xy}^2$$

$$= 2 \cdot 2 - a^2$$

So the point doesn't matter. D is positive, and f_{xx} is positive, unless $a^2 > 4$, i.e. $a > 2$ or $a < -2$. So we have a min for a between -2 and 2, and a saddle point otherwise.