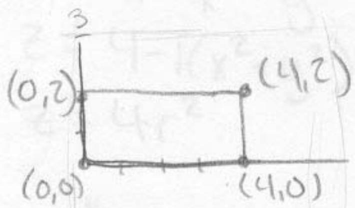
Exam 2 **Calc 3** **11/3/2006**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an iterated integral for the volume of the part of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle with vertices (0,0), (4,0), (0,2), and (4,2).

$y^2 + z^2 = 9$



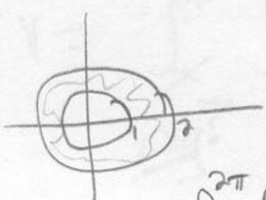
$$\int_0^4 \int_0^2 \sqrt{9-y^2}\, dy\, dx$$

Good!

2. Set up an iterated integral for the **surface area** of the portion of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$z = y^2 - x^2$

$SA = \iint \sqrt{f_x(x,y) + f_y(x,y) + 1}$



$SA = \iint \sqrt{(-2x)^2 + (2y)^2 + 1}\; dA$ $\sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2+1}$

$$\int_0^{2\pi} \int_1^2 \sqrt{4r^2+1}\; r\, dr\, d\theta$$

Great

3. Set up an iterated integral **in cylindrical coordinates** for the volume of the region between the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

convert to polar

$z=3(x^2+y^2)$

$z=3r^2$ $\quad z=4-r^2$

$3r^2=4-r^2$
$+r^2 \quad +r^2$

$\frac{4r^2}{4}=\frac{4}{4}$

$r^2=1 \quad r=1$

$0\le\theta\le 2\pi$

$0\le r\le 1$

$3r^2\le z\le 4-r^2$

$$\boxed{\int_0^{2\pi}\int_0^1\int_{3r^2}^{4-r^2} r\,dz\,dr\,d\theta}$$

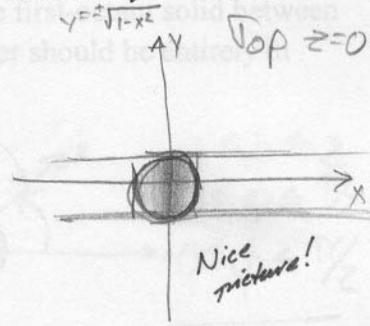
Good

4. Set up an iterated integral for the volume of the region inside both the cylinder $x^2 + y^2 = 1$ and the cylinder $y^2 + z^2 = 4$.

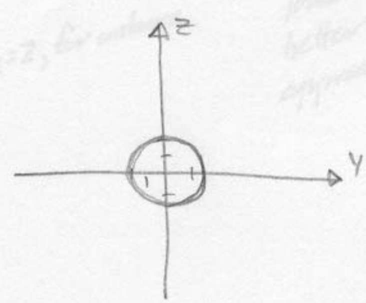
$z=\pm\sqrt{4-y^2}$

$$\boxed{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 1\;dz\,dy\,dx}$$

Great



Nice picture!



5. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$.

Integrand of 1 means volume!

It's one quarter of a sphere with radius 1, so

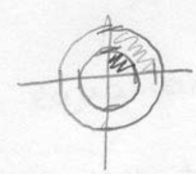
$$\frac{1}{4} \cdot \frac{4}{3}\pi \cdot 1^3 = \frac{\pi}{3}$$

6. Set up iterated integral(s) for the z-coordinate of the center of mass of the first-octant solid between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ [Note: For full credit, your final answer should be entirely in terms of a single coordinate system].

$x^2+y^2+z^2=1$
sphere w/ radius 1

$x^2+y^2+z^2=4$
sphere w/ radius 2

top view



use spherical coordinates

$z=\rho\cos\phi$ $\quad 1 \le \rho \le 2 \quad 0 \le \phi \le \pi/2 \quad 0 \le \theta \le \pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 z\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho\cos\phi\,\rho^2\sin\phi$$

$$\boxed{\bar{z} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta}{\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}}$$

Excellent!

7. Biff is a Calc 3 student at Enormous State University, and he's having some trouble. Biff says "Okay, dude, this is insane. Our calc professor just keeps saying these things like 'You should understand...' stuff, like we're supposed to be able to figure something out for ourselves or something. It's totally unfair. He said we should be able to understand why this one formula from Calc 2 was, like the one for $\overline{y} = \dfrac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$. But that's obviously a total crock, right, 'cause it's that way in the book. It's not like you figure out calculus yourself or anything, geez!"

   Explain clearly to Biff how to formula he mentions can be obtained from our Calc 3 approach.

Biff - look. ~~So~~, first rewrite that equation into something more familiar:

$$\overline{y} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{A}$$

. Now, look at the top of that equation. Doesn't it look like $\frac{1}{2}(y)^2$, which is what you get if you integrate y? Yes! In other words, it's almost as if someone's already done the inside part of a double integral. So, let's rewrite it as the original double integral.

$$\overline{y} = \frac{\int_a^b \int_0^{f(x)} y\,dy\,dx}{A}$$

See - if we were to work out the top, we'd get the top part from the calc II equation. Now, remember how the area is found with a double integral with an integrand of one? So let's do that:

$$\overline{y} = \frac{\int_a^b \int_0^{f(x)} y\,dy\,dx}{\int_a^b \int_0^{f(x)} 1\,dy\,dx}$$

The limits of integration come from the boundaries of whatever area we're using. and they match the ones on the top.

So, does this look familiar now? It's the $\overline{y} =$ formula we've been using! They just simplify it for Calc II so they don't have to learn double integrals.

Wonderful!

$$\frac{\cancel{a^2}u^2}{\cancel{a^2}} + \frac{\cancel{b^2}v^2}{\cancel{b^2}} + \frac{\cancel{c^2}w^2}{\cancel{c^2}} = 1$$

$$u^2 + v^2 + w^2 = 1$$

8. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation $x = au$, $y = bv$, $z = cw$.

$$\begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \\ \frac{dx}{dw} & \frac{dy}{dw} & \frac{dz}{dw} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \underline{abc}$$

$$= \int_0^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} 1\ abc \quad dw\, dv\, du$$

$u^2 + v^2 + w^2 = 1$ which we know is a sphere

with radius w (up+down) and $u + v$ (side-to-side respectively)

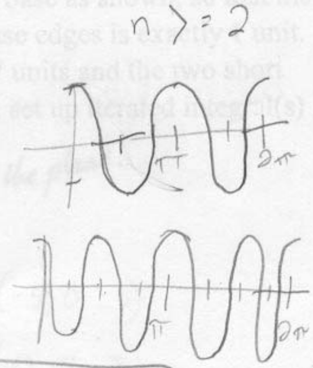
In class we know $V_{sphere} = \frac{4}{3}\pi r^3$

so $V = \frac{4}{3}\pi(abc)$ Yes.

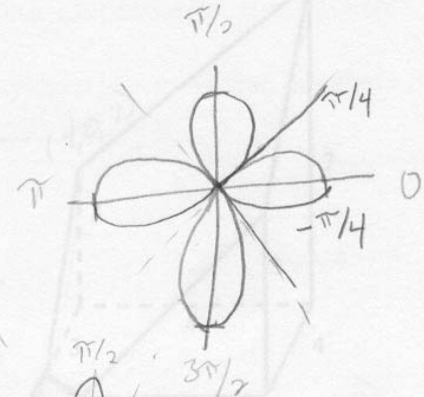
9. Set up an iterated integral for the area of one leaf of the rose $r = \cos n\theta$ where n is an integer greater than or equal to 2.

$$\int_{-\pi/4}^{\pi/4}\int_0^{\cos n\theta} r\,dr\,d\theta \qquad 4=2n \quad n \geq 2 \quad n=2$$

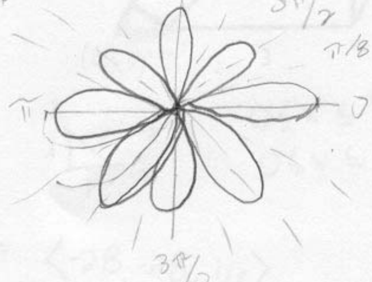
$\cos 2\theta$



$\cos 4\theta$



$$\boxed{\int_{-\pi/2n}^{\pi/2n}\int_0^{\cos n\theta} r\,dr\,d\theta}$$



$$\int_{-\pi/8}^{\pi/8}\int_0^{\cos n\theta} r\,dr\,d\theta$$

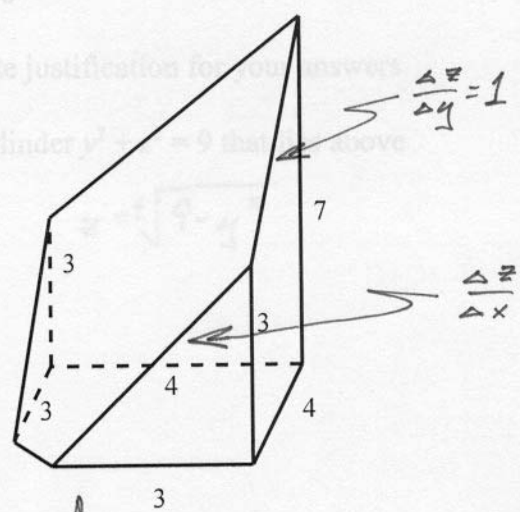
$8 = 2n$

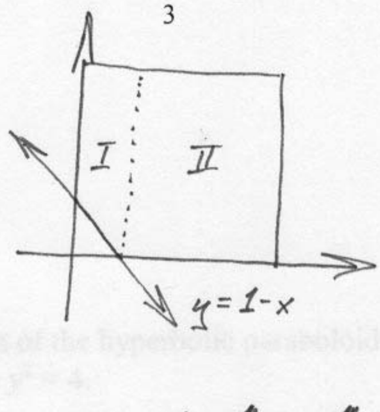
$n = 4$

Nice.

10. Consider the solid created by diagonally slicing a box with a square base of side length 4 with a plane that cuts off one corner of the base as shown, so that the length cut off from the base edges is exactly 1 unit. If the long vertical edge is 7 units and the two short vertical edges are 3 units, set up iterated integral(s) for the volume of this solid.

$$\underbrace{\int_0^1\int_{1-x}^4\int_0^{x+y-1} 1\,dz\,dy\,dx}_{\text{Part I}} + \underbrace{\int_1^4\int_0^4\int_0^{x+y-1} 1\,dz\,dy\,dx}_{\text{Part II}}$$





$z = -1 + x + y$