Exam 2 Real Analysis 1 11/10/2006

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. a) State the definition of continuity of a function at $x = a$.

Let $f: D \to R$ $D \subseteq R$ f is continuous at $a \in D$ iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x-a| < \delta$ and $x \in D$ implies $|f(x) - f(a)| < \varepsilon$.

Great

b) State the definition of continuity of a function on a set D.

f is continuous on a set D iff f is continuous at every point $a \in D$.

2. a) State the definition of the derivative of a function at $x = a$.

A function $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ and a is an accumulation point of D $(a \in D)$ has the derivative at $x = a$ defined by $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided the limit exists, $x \in D$.

Excellent

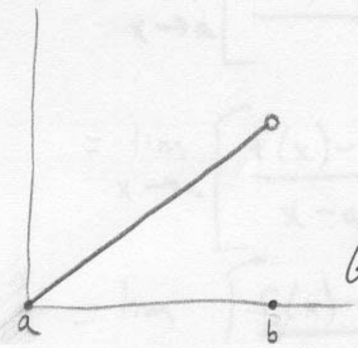
b) State the definition of differentiability of a function on a set D.

A function $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ is differentiable on the set D iff for every point in the set D, f is differentiable at that point.

3. Give an example of a function that meets two of the three assumptions of Rolle's Theorem but does not satisfy the conclusion of that theorem.

<u>assumptions</u>

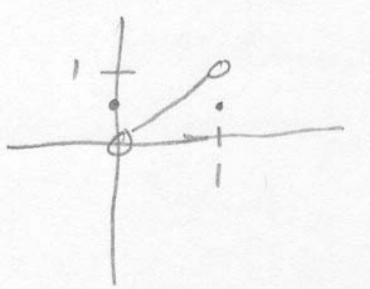
- f is continuous on $[a,b]$
- f is differentiable on (a,b)
- $f(a) = f(b)$



$f(a) = f(b)$ and f is differentiable on (a,b), but there is not $c \in (a,b) \ni f'(c) = 0$.

Good

4. Give an example of a function $f: [a, b] \to \mathbb{R}$ whose range is an open and bounded interval.



$$f(x) = \begin{cases} x & \text{for } x \in (0,1) \\ 1/2 & \text{for } x = 0 = 1. \end{cases}$$

Excellent

5. State and prove the Difference Rule for derivatives.

If f & g are differentiable, then $(f-g)'(x) = f'(x) - g'(x)$

proof $(f-g)'(x) = \lim\limits_{h \to 0} \dfrac{(f-g)(x+h) - (f-g)(x)}{h} = \lim\limits_{h \to 0} \dfrac{f(x+h) - g(x+h) - f(x) + g(x)}{h}$ regroup

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} = f'(x) - g'(x) \text{ as desired. } \blacksquare$$

f is differentiable so $= f'(x)$

g is differentiable so $= g'(x)$

Well done

6. State and prove the Boundedness Theorem.

If f be a function which is continuous on a bounded and closed set D, then f is bounded on D.

Proof: Suppose f is not bounded on D. Then, we can construct a sequence $\{x_n\}$ such than $|f(x_n)| > n$. Then by Bolzuano-W Theorem for sequence we can say that $\{x_n\}$ has subsequence $\{x_{n_k}\}$ which converge to c. $c \in D$ as D is closed. f is continuous on $c \in D$. Then, by prob set 8 prob 1 we can say that $\lim_{n \to \infty} \{f(x_{n_k})\} = f(c)$. which contradicts the fact that $|f(x_{n_k})| > n_k$. Hence, by contradiction f is bounded on D.

Excellent

Prob set 8 Prob 1 say

f is continuous on set $[a,b]$ and $\{x_n\}$ is a sequence converging to c. Then $f(x_n)$ converges to $f(c)$. where $c \in [a,b]$.

7. State and prove the Mean Value Theorem.

If f is continuous on $[a,b]$ & differentiable on (a,b) then $\exists\, c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof:

Let, $g(x) = f(x) - \frac{f(b)-f(a)}{b-a} \cdot (x-a)$

So, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \cdot 1$

From Rolle's theorem, we know $\exists\, c \in (a,b) \ni g'(c) = 0$

$\therefore\ g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$

$\therefore\ f'(c) = \frac{f(b)-f(a)}{b-a}$

Excellent.

8. Let E be an open subset of $\mathbb{R}$. Prove that $\mathbb{R} - E$ is closed.

Well, s'pose $\mathbb{R}-E$ wasn't closed, so didn't contain at least one of its limit points - Let's call that point a. Then a must be in E, and since E is open, there exists at least one neighborhood of a which lies entirely in E. But this contradicts the fact that a was a limit point of $\mathbb{R}-E$, so $\mathbb{R}-E$ must be closed after all. $\square$

$\Rightarrow b \le a - 1 \Rightarrow b < a$

9. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and that $a, b \in \mathbb{R}$ with $a - b \ge 1$. Does there have to exist a $c \in \mathbb{R}$ for which $f(c) = \dfrac{f(a) + f(b)}{2}$? Why or why not?

Yes, there does. Notice that $[b, a]$ is an interval on which f is continuous (since it's differentiable there), and (b, a) is an interval on which f is differentiable (since it's differentiable everywhere). Then either $f(a) \ne f(b)$, in which case since $\frac{f(a)+f(b)}{2}$ is between $f(a)$ and $f(b)$ and thus there exists such a c by the Intermediate Value Theorem, or $f(a) = f(b)$, in which case $\frac{f(a)+f(b)}{2} = \frac{f(a)+f(a)}{2} = \frac{2f(a)}{2} = f(a)$ and thus letting $c = a$ provides one such value of c. In either case such a c must exist. □

10. Determine whether the function $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable when x=0.

Well, by definition

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x}}{x}$$

$$= \lim_{x \to 0} x \sin\tfrac{1}{x}.$$

Now for all $x \in \mathbb{R}^+$, $-x \le x \sin\frac{1}{x} \le x$, and since $\lim_{x \to 0} x = 0$ and $\lim_{x \to 0} -x = 0$, we can conclude by the Squeeze Theorem that $\lim_{x \to 0} x \sin\frac{1}{x} = 0$ as well. Then since this limit exists, $f(x)$ is differentiable when $x = 0$. □