Exam 1 Calc 1 9/21/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the definition of the derivative of the function $f(x)$ at the point $x = a$.

$$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Good

Use the graph of $g(x)$ at the bottom of the page for problems 2 and 3:

2. Find the following limits:

a) $\lim_{x\to -3^+} g(x)$ $\quad \lim_{x\to -3^+} g(x) = 2$

b) $\lim_{x\to -3} g(x)$ $\quad \lim_{x\to -3} g(x) = 2$

c) $\lim_{x\to 1} g(x)$ $\quad \lim_{x\to 1} g(x) = 1$

Great!

d) $\lim_{x\to 2^-} g(x)$ $\quad \lim_{x\to 2^-} g(x) = 2$

e) $\lim_{x\to 2} g(x)$ $\quad \lim_{x\to 2} g(x)$ Limit does not exist $\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$

3. a) For which values of x does the function fail to be continuous?

$x = -3, x = 1, x = 2$

b) For which values of x does the function fail to be differentiable?

$x = -3$: not continuous

$x = 0$: come to a corner

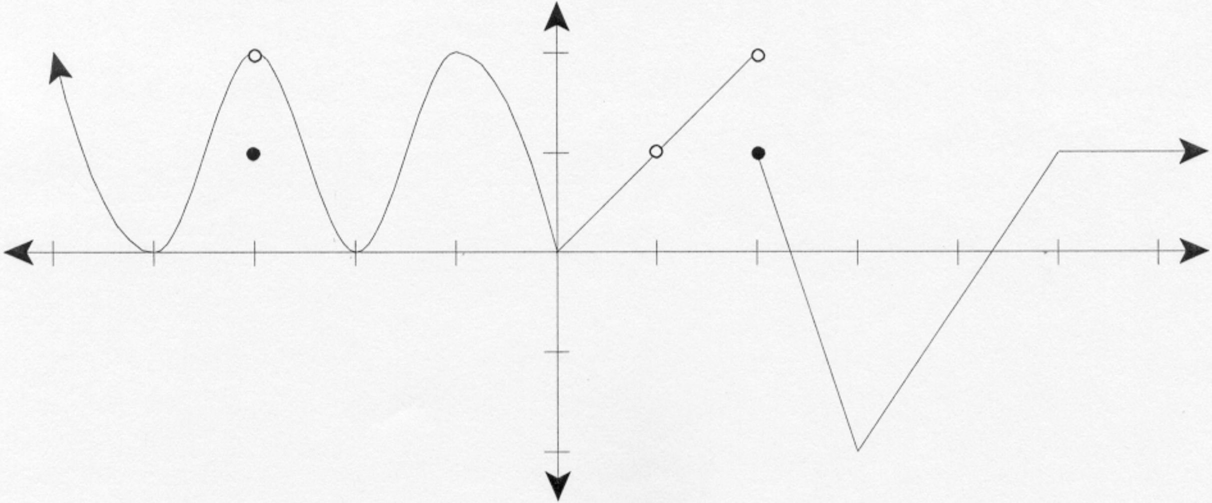
$x = 1$: not continuous

$x = 2$: not continuous

$x = 3$: comes to a corner

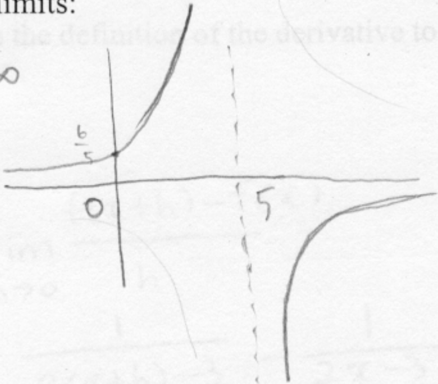
$x = 5$: comes to a corner.

Wonderful!



4. Evaluate the following limits:

a) $\lim_{x \to 5^+} \frac{6}{5-x}$ $= -\infty$



Nice!

b) $\lim_{x \to \infty} \frac{6}{5-x}$. $= 0$

5. Evaluate $\lim_{h\to 0}\dfrac{(3+h)^2-9}{h}$. $= \lim_{h\to 0}\dfrac{\cancel{9}+6h+h^2\cancel{-9}}{h}$

$= \lim_{h\to 0}\dfrac{\cancel{h}(h+6)}{\cancel{h}} = 0+6 = \boxed{6}$

Excellent!

The limit as h approaches 0 is 6.

6. If $f(x)=\dfrac{1}{2x-3}$, use the definition of the derivative to find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{2(x+h)-3} - \dfrac{1}{2x-3}}{h}$$

$$= \lim_{h\to 0} \frac{2x-3-2(x+h)+3}{\{2(x+h)-3\}\cdot\{2x-3\}} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x}-\cancel{3}-\cancel{2x}-2h+\cancel{3}}{\{2(x+h)-3\}\{2x-3\}} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\{2(x+h)-3\}\{2x-3\}} \times \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-2}{\{2(x+h)-3\}\{2x-3\}}$$

$$= \frac{-2}{(2x-3)(2x-3)}$$

$$= \frac{-2}{(2x-3)^2}$$

Well done!

7. Evaluate $\lim_{x\to 0}\left(\frac{\sin(2+x)-\sin 2}{x}\right)$ to the nearest 0.001.

When $x=.1$, $\frac{\sin(2+.1)-\sin 2}{.1} \approx -.460881$

" $x=.01$, $\frac{\sin(2+.01)-\sin 2}{.01} \approx -.420686$

" $x=.001$, $\frac{\sin(2+.001)-\sin 2}{.001} \approx -.416601$

" $x=.0001$, $\frac{\sin(2+.0001)-\sin 2}{.0001} \approx -.416151$

" $x=.00001$, $\frac{\sin(2+.00001)-\sin 2}{.00001} \approx -.416147$

So to the nearest thousandth, it seems to stabilize at -0.416.

Doublecheck from the left:

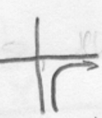
When $x=-.0001$, $\frac{\sin(2-.0001)-\sin 2}{-.0001} \approx -.416101$

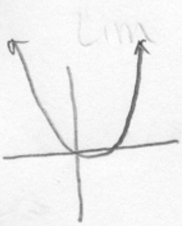
Yep.

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, I'm gonna have to drop Calc and my dad's gonna kill me for it. Our professor is off the deep end, though, and like half the class failed the first test. He said we deserved to fail, though, because of what this one guy in the class wrote for one question. Like, he wrote something about how you know the limit of a function is infinity when the height keeps getting bigger and bigger. The professor was saying all this stuff about how sometimes the heights get bigger and bigger without being infinity, but he pretty much lost me. I mean, it seemed to me like if every time you make *x* bigger, then the *y* gets bigger too, then that's when you say the limit is infinity, right? So anyway, I guess I'll drop out and flip burgers."

Help Biff by explaining as clearly as you can what it means to say that the limit of a function (either as *x* approaches some fixed value *a* or as *x* approaches infinity) is infinity.

$$\lim_{x\to\infty} f(x)$$

Biff, I understand your frustration. Let me try to explain it to you. The limit of a function, for example $f(x) = \lim_{x\to\infty} \frac{x^2}{3}$ is approaching infinity because the higher the x values that you plug into the equation, the higher the y values. However, sometimes like your instructor told you the height of a function gets bigger and bigger w/o being infinity. If you have a function that looks like this , you can see that the height (y-values) keep getting bigger, however, they are not approaching infinity but instead 0.



In the graph shown above, you can see that x is getting bigger and y is getting bigger, and that the graph is approaching 0. You should always try to plug in numbers, if you are not sure if a fkt is approaching infinity or not. The numerical approach is always very helpful.

Excellent!

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9. Evaluate $\lim_{x \to \infty}\left(x - \sqrt{x^2 + ax + b}\right)$, where a and b are constants.

$$\lim_{x \to \infty} \frac{x - \sqrt{x^2+ax+b}}{1} \cdot \frac{x + \sqrt{x^2+ax+b}}{x+\sqrt{x^2+ax+b}}$$

$$\lim_{x \to \infty} \frac{\cancel{x^2} - \cancel{x^2} - ax - b}{x+\sqrt{x^2+ax+b}}$$

$$\lim_{x \to \infty} \frac{-ax-b}{x+\sqrt{x^2+ax+b}}$$

$$\lim_{x \to \infty} \frac{-a-\frac{b}{x}}{1+\sqrt{1+\frac{a}{x}+\frac{b}{x^2}}}$$

$$\lim_{x \to \infty} \frac{-a-0}{1+\sqrt{1+0+0}}$$

Wonderful!

$$\lim_{x \to \infty} = \frac{-a}{2}$$

10. Suppose that $f(x)$ is a function whose derivative exists, and that $g(x) = f(x) + C$, where C is some constant. What can you say about $g'(x)$, and why?

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

since $g(x) = f(x) + C$

$\rightarrow g(x+h) = f(x+h) + C$

$$\rightarrow g'(x) = \lim_{h \to 0} \frac{f(x+h) + C - [f(x) + C]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cancel{+C} - f(x) \cancel{-C}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

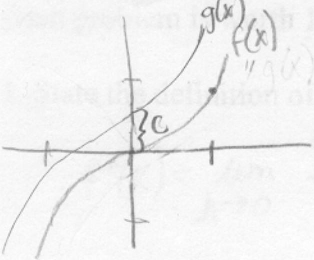
Outstanding!

according to the derivative definition:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

So: $\boxed{g'(x) = f'(x)}$

10. Suppose that $f(x)$ is a function whose derivative exists, and that $g(x) = f(x) + C$, where C is some constant. What can you say about $g'(x)$, and why?



Because f(x) has a derivative and g(x) is just f(x) shifted up the value of C we can say that $g'(x)$ is the same as $f'(x)$. The derivative is the slope of the line. By adding c to f(x) you do not change the slope. Therefore we can say $g'(x) = f'(x)$.

Nice.