Exam 2 Calc 1 10/12/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. What is the derivative of $y = e^x + 5\tan x - 4\pi$?

$y = e^x + 5\tan x - 4\pi$

$y' = e^x + 5\sec^2 x - 0 = \boxed{e^x + 5\sec^2 x}$

Exponential rule

constant Multiple rule

constant rule

Excellent!

2. a) If $g(x) = x \cdot f(x)$, what is $g'(x)$?

$g(x) = x \cdot f(x)$

$g'(x) = x \cdot f'(x) + 1 \cdot f(x)$

$g'(x) = x f'(x) + f(x)$ ← product rule

Great!

b) If $h(x) = \left[f(x) \right]^3$, what is $h'(x)$?

$h(x) = [f(x)]^3$

$h'(x) = 3[f(x)]^2 \cdot f'(x)$ ← chain rule

3. Let $f(x) = C$, where C is some constant. Prove that $f'(x) = 0$.

Constant Rule

$f(x) = C$ then $f'(x) = 0$

Proof:

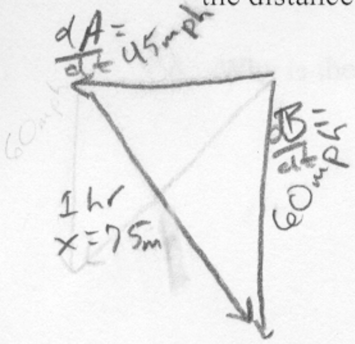
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{C - C}{h}$$

$$= \lim_{h \to} \frac{0}{h} = 0 = f'(x)$$

Excellent!

4. Two cars leave an intersection at the same time, with one traveling West at 45 miles per hour and the other traveling South at 60 miles per hour. How fast, to the nearest mile per hour, is the distance between the two cars changing two hours later?



2 hrs

Know: $a^2 + b^2 = c^2$

Differentiate:

$$2a \cdot \frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(90)(45) + 2(120)(60) = 2(150)\frac{dc}{dt}$$

$$8100 + 14400 = 300\frac{dc}{dt}$$

$$\frac{22500}{300} = \frac{dc}{dt}$$

$$\boxed{75\text{mph} = \frac{dc}{dt}}$$

Well done

$45^2 + 60^2 = x^2$

$x = \sqrt{5625}$

$x = 75$

At 2 hrs

A = 90 mi

B = 120 mi

C = 150 mi

5. State and prove the Product Rule for derivatives. Make it clear how you use any assumptions.

Product Rule: If $f(x)$ and $g(x)$ are differentiable, then $(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$.

Proof: Well,

$$(f\cdot g)'(x) = \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\cdot g(x+h) + \lim_{h\to 0} f(x)\cdot\frac{g(x+h) - g(x)}{h}$$

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x). \quad \square$$

This only holds if all these limits exist

These limits exist because f and g are differentiable.

~~6~~. Why is the derivative of ln x equal to $1/x$?

I know: $e^{(\ln x)} = x$

derivative: $e^{(\ln x)} \cdot (\ln x)' = 1$

$x \cdot (\ln x)' = 1$

$\boxed{(\ln x)' = \frac{1}{x}}$

Nice!

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

7. Why is the derivative of cos *x* equal to –sin *x*?

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos x = 0 \quad 1$

$$f(x) = \cos x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1)}{h} + \frac{-\sin x \sin h}{h}$$

Sum Rule, Product Rule for Limits

$$= \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{approaches } 0} + -\sin x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{approaches } 1}$$

$$= -\sin x.$$

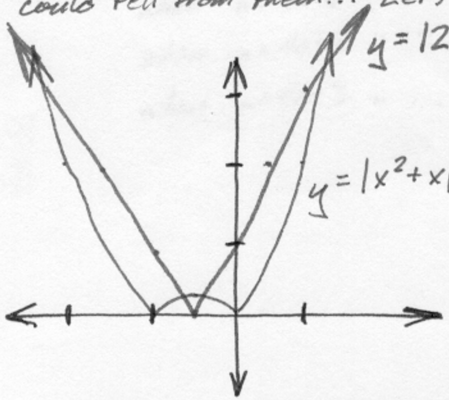
Beautiful!

8. Bunny is a calculus student at Enormous State University, and she's having some trouble. "Ohmygod, I like, have no chance of even surviving calculus. It's, like, literally killing me. I thought I pretty much knew calculus from my AP class in high school, right? But, like, now it's all different stuff. We got, like, this study guide for our test? Like with 40 problems on it? And like, they say that a bunch of those will be on the test, so you have to know how to do them, right? But one of the questions is, like, whether the derivative of the absolute value of x squared plus x is equal to the absolute value of $2x$ plus 1. But that's so unfair, because in AP we never did the absolute thingies. I think I'm going to tell Daddy and see if he can get the professor fired."

Help Bunny by explaining as clearly as possible how she might determine whether the derivative she mentions is correct (just in case her plan to get the professor fired fails).

Well Bunny, I think first of all that there's not really a nice formula for the derivative of the absolute value function – remember from before that it's not even differentiable when $x=0$?

So instead of using formulas, you'll have to think about it some other way. I'd look at the graphs of $y=|x^2+x|$ and $y=|2x+1|$ and see what I could tell from them... Let's see what your calculator says...



So Bunny, if the green graph in the picture were the derivative of the blue graph, it would have positive heights at all the x values where the blue graph has positive slopes. But the height of the green graph is positive everywhere, so the blue graph would have to have positive slopes everywhere, which it definitely doesn't. So whatever you might be able to say about the derivative of $|x^2+x|$, it definitely isn't $|2x+1|$.

9. Let $g(x) = e^x \sin x$.

a) What is $g'(x)$? Product Rule!

$$g'(x) = e^x \cdot \sin x + e^x \cdot \cos x$$

b) What is $g^{(n)}(x)$?

$$g''(x) = e^x \cdot \sin x + e^x \cdot \cos x + -e^x \sin x + e^x \cos x = 2e^x \cos x$$

$$g'''(x) = 2e^x \cdot \cos x + -2e^x \cdot \sin x$$

$$g^{(4)}(x) = 2e^x \cos x - 2e^x \sin x - 2(e^x \sin x + e^x \cos x) = -4e^x \sin x$$

But that's -4 times $g(x)$, so its derivative will be -4 times $g'(x)$, etc.

So when $n = 4a$ for a natural number a, $g^{(n)}(x) = (-4)^a \cdot e^x \sin x$

when $n = 4a+1$ " " " " ", $g^{(n)}(x) = (-4)^a(e^x \sin x + e^x \cos x)$

when $n = 4a+2$ " " " " ", $g^{(n)}(x) = (-4)^a \cdot 2e^x \cos x$

when $n = 4a+3$ " " " " ", $g^{(n)}(x) = (-4)^a \cdot 2e^x(\cos x - \sin x)$

10. Find both points on the ellipse $x^2 + 4y^2 = 36$ whose tangent lines pass through the point (12,3).

take the derivative of $x^2+4y^2=36$

$$2x + 8y \cdot y' = 0$$

$$8y y' = -2x$$

$$y' = \frac{-2x}{8y} = \frac{-x}{4y}$$ Yes!

Let's say point (a,b) is on the $x^2+4y^2=36$

$$y - b = -\frac{a}{4b}(x-a)$$ tangent line equation

This line passes (12,3)

$$3 - b = \frac{-a}{4b}(12-a)$$

Wonderful!

$$4b(3-b) = -a(12-a)$$

$$12b - 4b^2 = -12a + a^2$$

$$12b + 12a = a^2 + 4b^2 = 36$$

$$12b + 12a = 36$$

$$b + a = 3$$

This should satisfy $a^2 + 4b^2 = 36$

$$(3-b)^2 + 4b^2 = 36$$

$$9 - 6b + b^2 + 4b^2 = 36$$

$$5b^2 - 6b - 27 = 0$$

$5 \qquad 9$

$$(5b+9)(b-3) = 0$$

$$b = 3, -\frac{9}{5}$$

$$a = 0, \ 4.8$$

$$\underline{\underline{(0,3) \ (4.8, -\tfrac{9}{5})}}$$