Exam 3 Calc 1 11/9/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\lim_{x\to\infty} \frac{e^x}{x^3}$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

Great!

2. Given the following information about a continuous function g(x), determine the intervals of increase and decrease and intervals of concavity of $g(x)$

	$(-\infty,-3)$	$(-3,0)$	$(0,5)$	$(5,+\infty)$
$g'(x)$	+	+	–	–
$g''(x)$	+	–	–	+

It's increasing where the first derivative is positive, so

$(-\infty,-3) \cup (-3,0)$

and decreasing where the first derivative is negative, so

$(0,5) \cup (5,+\infty)$

It's concave up where the second derivative is positive, so

$(-\infty,-3) \cup (5,+\infty)$

and concave down where the second derivative is negative, so

$(-3,0) \cup (0,5)$.

max + mins

$-1 \quad +2 \quad -1$

3. Let $f(x) = x^3 + 2x^2 - 1$. Find the absolute extrema of f on the interval $[-1, 1]$.

$f(x) = x^3 + 2x^2 - 1$

$f'(x) = 3x^2 + 4x$

$0 = x(3x + 4)$

$x = 0 \quad x = -4/3$ ← eliminate b/c it's out of the interval from $[-1, 1]$

$f(1) = 2$

$f(0) = -1$

$f(-1) = 0$

Absolute max when $x = 1$

+

Absolute min when $x = 0$

Excellent!

4. Suppose that a company's cost function is $C(x) = 40{,}000 + 300x + x^2$. Find the production level that will minimize the average cost.

$$\text{average cost} = \frac{C(x)}{x} = \frac{40{,}000}{x} + 300 + x$$

$$\text{Marginal cost} = c'(x) = 300 + 2x$$

The production level that will minimize the averge cost:

$$\frac{c(x)}{x} = c'(x)$$

$$\Leftrightarrow \frac{40{,}000}{x} + \cancel{300} + x = \cancel{300} + 2x$$

$$\Leftrightarrow x - \frac{40000}{x} = 0 \Leftrightarrow \frac{x^2 - 40{,}000}{x} = 0$$

$$\Leftrightarrow x^2 - 40{,}000 = 0$$

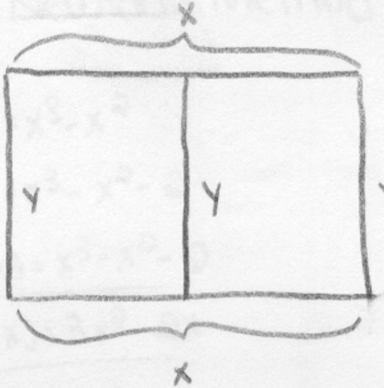
$$\Leftrightarrow x^2 = 40000$$

$$x = 200$$

So the average cost minimizes when $x = 200$

Excellent!

8. A farmer wants to create a rectangular lot for his emu herd. He has 4800 feet of fence, and wants to have a dividing fence down the middle of the lot (parallel to one of the outside edges) to keep the males and females separated. What is the largest area that can be created?



$4800 = x + x + y + y + y = 2x + 3y$

$4800 - 2x = 3y$

$y = \dfrac{4800 - 2x}{3}$

$A(x) = x \cdot y$

$A(x) = x \cdot \left(\dfrac{4800 - 2x}{3}\right)$

$= \dfrac{4800}{3}x - \dfrac{2}{3}x^2$

$= 1600x - \frac{2}{3}x^2$

$A'(x) = 1600 - \frac{4}{3}x$

$0 = 1600 - \frac{4}{3}x$

$\frac{4}{3}x = 1600$

$4x = 4800$

$x = 1200$

Well done!

$A(x) = 1600x - \frac{2}{3}x^2$

$A(1200) = 1600(1200) - \frac{2}{3}(1200)^2$

$= 1920000 - 960000$

$= 960000$

$\therefore$ largest area $= 960000 \text{ ft}^2$

6. Use Newton's Method with an initial approximation of $x_1 = 2$ to find x_2, the second approximation to a solution of the equation $2 = x^3 - x^2$.

$2 = x^3 - x^2$

so $f(x) = x^3 - x^2 - 2$

and $f'(x) = 3x^2 - 2x$

Newtons method: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(2) = 2^3 - 2^2 - 2 = 8 - 4 - 2 = 2$

$f'(2) = 3(2)^2 - 2(2) = 8$

$x_2 = 2 - \frac{f(2)}{f'(2)}$

$x_2 = 2 - \frac{2}{8}$

$x_2 = 2 - \frac{1}{4}$

$\boxed{x_2 = 1.75}$

Good

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Well, crap. Our professor is gonna give us this test that's mostly true/false, 'cause he said he has lots of more important things to do than waste time grading our stuff, and so I thought that was cool at first 'cause I figured true/false would be easy, right? But then this girl in the class who sits by me, and I guess she studies a lot or something, she said she got a copy of a true/false Calc exam from last year or something, and I had no idea what was up with a lot of them. One was, like, if a function could have a max at a place where the derivative wasn't zero. I figured that one I knew for sure was false, 'cause if there's one thing I learned it's that you gotta take the derivative and set it equal to zero, right? But she said from the key it was true. How the heck could that be?"

Help Biff understand how a function could have a maximum value at a point where the derivative is non zero.

Biff, there are actually two ways a function can have a maximum and the derivative does not equal 0. When you are looking to find the maximums and minimums, you have to take the derivative and see where it is equal AND see where it is undefined. for example. if you have the equation $f(x) = \frac{4x}{x^3}$ the derivative $= \frac{4x^3 - (4x \cdot 3x^2)}{(x^3)^2}$

So, there are critical numbers where this equals 0 and at $x = 0$ because that is where this equation is undefined.

The other place it can happen is just like number 3 on this test. If the problem states an interval, then you have to try both end points to see if they happen to be the maximum. So Biff, as you can see, there are always exceptions to everything in Calculus. *Excellent!*

8. Find the exact x-coordinates of the global extrema of $f(x) = \dfrac{x-3}{x^2+1}$.

$$f(x) = \frac{x-3}{x^2+1}$$

$$f'(x) = \frac{(x^2+1) - (x-3)(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{x^2+1-2x^2+6x}{(x^2+1)^2}$$

$$f'(x) = \frac{-x^2+6x+1}{(x^2+1)^2}$$

$$(x^2+1)^2 \cdot \quad 0 = \frac{-x^2+6x+1}{\cancel{(x^2+1)^2}} \cdot \cancel{(x^2+1)^2}$$

Quadratic Formula:

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$0 = -x^2+6x+1$$

$$\cancel{0 = -1(x^2+6x-1)}$$

$$\cancel{(x-1)(x+1)}$$

$$\text{so, } \frac{-6 \pm \sqrt{(6)^2-4(-1)(1)}}{2(-1)}$$

$$= \frac{-6 \pm \sqrt{36+4}}{-2}$$

40 → 2 · 20 → 2 · 10 → 2 · 5

$2\sqrt{10}$

Well done!

$$= \frac{-6 \pm \sqrt{40}}{-2}$$

$$= \frac{-6}{-2} + \frac{\cancel{2}\sqrt{10}}{\cancel{-2}} \quad \text{OR} \quad \frac{-6}{-2} - \frac{2\sqrt{10}}{-2}$$

$$\boxed{x = 3 - \sqrt{10} \quad \text{OR} \quad 3 + \sqrt{10}}$$

9. a) Let $f(x) = x^3 - 3x^2 + 2x$. Find all inflection points of f.

b) Suppose that f is a third degree polynomial which crosses the x-axis at three distinct point x_1, x_2, and x_3. Show that f must have an inflection point at $(x_1 + x_2 + x_3)/3$.

a, $f'(x) = 3x^2 - 6x + 2$

$f''(x) = 6x - 6$

$f''(x) = 0$ when $6x - 6 = 0 \Rightarrow \underline{x = 1}$ Good.

$\Rightarrow y = 1 - 3 + 2 = 0$.

The inflection point of x is $(1, 0)$

b, $f(x) = (x - x_1)(x - x_2)(x - x_3)$

$f'(x) = (x - x_2)(x - x_3) + (x - x_1)(x - x_2) + (x - x_1)(x - x_3)$

$f''(x) = x - x_2 + x - x_3 + x - x_1 + x - x_2 + x - x_1 + x - x_3$

$= 6x - 2(x_1 + x_2 + x_3)$

So $f''(x) = 0$ when $6x = 2(x_1 + x_2 + x_3)$

$\Rightarrow x = \dfrac{x_1 + x_2 + x_3}{3}$

inflection point □

Nice Job!

10. Functions of the form $f(t) = ate^{-bt}$, for suitable values of constants a and b, can be used to model quantities such as the amount of caffeine in the bloodstream t minutes after drinking a caffeinated beverage. A study has determined that the maximum amount of caffeine in the bloodstream occurs 37.5 minutes after ingestion. Suppose that the peak amount of caffeine in a person's bloodstream is 50mg. What values for the constants a and b would produce a suitable function to model this situation?

$$f'(t) = a \cdot e^{-bt} + at \cdot e^{-bt} \cdot -b = ae^{-bt} - abte^{-bt}$$

$$0 = ae^{-bt}(1 - bt)$$ Now to have a max when $t = 37.5$,

$$0 = \underbrace{ae^{-b(37.5)}}_{\text{This can't be zero}} \underbrace{(1 - b(37.5))}_{\text{So this must be zero}}$$

$$1 - b \cdot 37.5 = 0$$

$$1 = b \cdot 37.5$$

$$\therefore b = \frac{1}{37.5}$$

So $f(t) = ate^{-\frac{1}{37.5}t}$, and we know $f(37.5) = 50$, so

$$(50) = a \cdot (37.5) e^{-\frac{1}{37.5}(37.5)}$$

$$50 = a \cdot 37.5 \cdot e^{-1}$$

$$\therefore a = \frac{50e}{37.5}$$