Exam 2 Calc 2 10/12/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Show that $\int x \cos x \, dx = x \sin x + \cos x + C$.

use integration by parts

let $u = x$ $\quad v = \sin x$

$u' = 1$ $\quad v' = \cos x$

$\int x \cos x \, dx$

$x \sin x - \int 1 \sin x \, dx$

$x \sin x - -\cos x + C$

Good

$\boxed{x \sin x + \cos x + C}$

2. Set up an integral for the arc length of the curve $y = \sin x$ from (0,0) to (π,0).

$$L = \int_0^{\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$y = \sin x$

$y' = \cos x$

$$\boxed{L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx}$$

Yep!

3. Set up an integral for the surface area of the solid formed by rotating the curve $y = \sin x$ from $(0,0)$ to $(\pi,0)$ around the x-axis.

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

$$2\pi \int_0^{\pi} \sin x \sqrt{1 + (\cos x)^2}\,dx$$

Yup.

4. The manager of a fast-service penguin vet clinic in Antarctica determines that the probability density function for the waiting time of patients at the clinic is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{10}e^{-t/10} & \text{if } t \geq 0 \end{cases}$.

a) Set up an integral for the probability that a patient has to wait more than 5 minutes.

$$(P>5) = \int_5^{\infty} \frac{1}{10} e^{-t/10}\, dt$$

b) The manager wants to start an advertising campaign where any patient who has to wait more than a certain amount of time receives all the herring they can eat for life, but doesn't want more than 1% of the patients to win the free herring. Set up an equation whose solution would give a suitable length of time for the guarantee.

prob. = .01

$$.01 = \int_t^{\infty} \frac{1}{10} e^{-t/10}\, dt$$

t = time

$\int_t^{\infty} \rightarrow$ past certain time

Excellent!

Great

5. Derive the integration formula $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$.

$$\int \tan^n u\,du = \int \tan^2 u \cdot \tan^{n-2} u\,du$$

$\sin^2 u + \cos^2 u = 1$

$\tan^2 u + 1 = \sec^2 u$

$\tan^2 u = \sec^2 u - 1$

$$= \int (\sec^2 u - 1)\cdot \tan^{n-2} u\,du$$

$$= \int \tan^{n-2} u \cdot \sec^2 u\,du - \int \tan^{n-2} u\,du$$

Let $v = \tan u$

$\frac{dv}{du} = \sec^2 u$

$\frac{dv}{\sec^2 u} = du$

$$= \int v^{n-2} \cdot \sec^2 u \cdot \frac{dv}{\sec^2 u} - \int \tan^{n-2} u\,du$$

$$= \int v^{n-2}\,dv - \int \tan^{n-2} u\,du$$

$$= \frac{v^{n-1}}{n-1} - \int \tan^{n-2} u\,du$$

$$= \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

6. Show that $\int \frac{\sqrt{u^2 - a^2}}{u}\,du$ can be converted to $\int \tan^2\theta\,d\theta$.

let $u = a\sec\theta$

$du = a\sec\theta\tan\theta$

$$\int \frac{\sqrt{(a\sec\theta)^2 - a^2}}{\cancel{a\sec\theta}} \cdot \cancel{a\sec\theta}\tan\theta\,d\theta$$

$$\int \frac{\sqrt{a^2(\sec^2\theta - 1)}}{1} \cdot \tan\theta\,d\theta$$

$$\int a\sqrt{\tan^2\theta} \cdot \tan\theta\,d\theta$$

$$a\int \tan\theta \cdot \tan\theta\,d\theta$$

Well done!

$$a\int \tan^2\theta\,d\theta$$

7. Biff is a calculus student at Enormous State University, and he has a question. Biff says "Dude, I'm cramming for my calc test, and I think something's wrong. I got notes from this friend of mine Bunny, 'cause I skipped class for, uh, like a week 'cause I was pretty busy, you know? And like, with the integrals for trig, you know? She's got in her notes about what you do when it's sin and cos, and she's got when it's tan and sec, but she doesn't have ones for the other pair. Heck, I don't even get why they come in pairs like that anyway, but do you know what I'm supposed to do for the other ones?"

Help Biff by explaining first "why they come in pairs like that", and second why it's reasonable that cot and csc don't show up in Bunny's notes.

They come in pairs like that because it is easier to find the integral for the function. If we had a function $\frac{\sin x}{\tan x}$, we would want to turn them into a "pair". The pairs are $\sin x$ and $\cos x$, $\sec x$ & $\tan x$ and $\csc x$ & $\cot x$; they all work well together because their derivatives can cancel each other out and let us do the integral. $\csc x$ & $\cot x$ don't show up very much because they can also be written in the form of $\frac{1}{\tan}$, which equals $\frac{\cos x}{\sin x}$ and csc can be written as $\frac{1}{\sin x}$. We like these functions a lot more than $\cot x$ & $\csc x$ so we always write them in terms of $\sin x$ & $\cos x$.

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Excellent!

ex $\int \cos^2 x \cdot \sin x$

$u = \cos x$
$du = -\sin x \, dx$

$\int u^2 \cdot \cancel{\sin x} \cdot \frac{du}{-\cancel{\sin x}}$

$\int u^2 \, du$

8. Derive the formula $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$.

$$\int \frac{1}{u(a+bu)} = \frac{A}{u} + \frac{B}{a+bu}$$

$$1 = A(a+bu) + Bu$$

if $u = 0$ $\quad 1 = aA$ $\quad A = 1/a$

$a+bu = 0$

$bu = -a$

$u = -a/b$

if $u = -a/b$

$1 = B(-a/b)$

$B = -b/a$

$$\int \frac{A}{u}\,du + \frac{1}{a}\int \frac{b}{a+bu}\,du$$

let $z = a+bu$

$dz = b\,du$

$$\frac{1}{a}\ln|u| - \frac{1}{a}\int \frac{1}{z}\,dz$$

$*\ \ln a - \ln b = \ln \frac{a}{b}$

$$\frac{1}{a}\left(\ln|u| - \ln|a+bu|\right) + C$$

$$\frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

Nice

9. Find the x coordinate of the center of mass of the first-quadrant portion of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

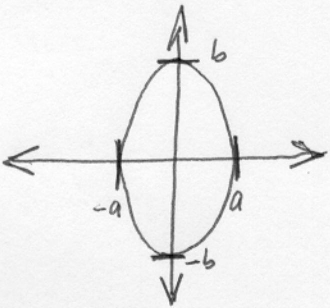
$$\Downarrow$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y = \pm\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$

And we'll use + for the first quadrant



So

$$\bar{x} = \frac{\int_0^a x\cdot\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\,dx}{\int_0^a \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\,dx}$$

$$= \frac{\frac{b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx}{\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx}$$

$$= \frac{\left[\frac{-1}{3}(a^2-x^2)^{3/2}\right]_0^a}{\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a}$$

By letting $u=a^2-x^2$

By Line 30

$$= \frac{0+a^3/3}{\left(0+\frac{a^2}{2}\sin^{-1}1\right)-(0+0)}$$

$$= \frac{a^3}{3} \div \frac{\pi a^2}{4}$$

$$= \frac{a^3}{3}\cdot\frac{4}{\pi a^2}$$

$$= \frac{4a}{3\pi}$$

10. Evaluate $\int_1^{\infty} \frac{\ln x}{x^3}\,dx$.

$$\lim_{b\to\infty} \int_1^b \ln x \; x^{-3}\,dx$$

$u = \ln x \qquad v = \frac{-1}{2x^2}$

$u' = \frac{1}{x} \qquad v' = x^{-3}$

$$\lim_{b\to\infty} \left[\frac{-\ln x}{2x^2} + \int \frac{1}{x}\cdot\frac{1}{2x^2}\,dx \right]$$

$$\lim_{b\to\infty} \left[\frac{-\ln x}{2x^2} - \frac{1}{4x^2} \right]_1^b$$

Wow!

$$\underbrace{\frac{-\ln b}{2b^2}}_{0} - \underbrace{\frac{1}{4b^2}}_{0} - \underbrace{\frac{-\ln|1|}{2}}_{0} + \frac{1}{4}$$

$$\boxed{\frac{1}{4}}$$