Exam 4 Calc 2 12/5/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the sum of the series $\frac{3}{4} - \frac{3}{10} + \frac{3}{25} - \frac{6}{125} + \frac{12}{625} - \ldots$

$S = \frac{a}{1-r}$

$a = \frac{3}{4}$

$r = -\frac{2}{5}$

$|r| < 1$

$S = \frac{\frac{3}{4}}{1-\left(-\frac{2}{5}\right)} \rightarrow \frac{\frac{3}{4}}{\frac{7}{5}}$

$\boxed{S = \frac{15}{28}}$

Great!

2. Give a power series for $f(x) = \frac{\sin x}{x}$ of at least 4th degree.

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$

Excellent!

$a = \frac{\pi}{2}$

3. Find a Taylor polynomial of degree at least 4 for $f(x) = \cos x$ centered at $x = \pi/2$.

$f(x) = \cos x \quad f(\frac{\pi}{2}) = 0$ $\qquad p(x) = f(x) + f'(x)(x-a) + \frac{f''(x)(x-a)^2}{2!}$

$f'(x) = -\sin x \quad f'(\frac{\pi}{2}) = -1$

$f''(x) = -\cos x \quad f''(\frac{\pi}{2}) = 0$

$f'''(x) = \sin x \quad f'''(\frac{\pi}{2}) = 1$ $\qquad$ Lovely.

$f^4(x) = \cos x \quad f^4(\frac{\pi}{2}) = 0$

$$\boxed{p(x) = -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5}$$

4. Determine whether $\sum_{n=1}^{\infty} \frac{3n}{n^3+1}$ converges or diverges.

Use limit Comparison w/ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{3n}{n^3+1}} \to \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^3+1}{3n} \to \lim_{n\to\infty} \frac{(n^3+1)\frac{1}{n^3}}{(3n^3)\frac{1}{n^3}}$$

$$\lim_{n\to\infty} \frac{1 + \frac{1}{n^3} \to 0}{3} \to \frac{1}{3}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series w/ $p > 1$, by the limit comparison test $\sum_{n=1}^{\infty} \frac{3n}{n^3+1}$ must also converge.

Wonderful!

5. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges or $\boxed{\text{diverges.}}$

* Use integral test

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$$\lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}}\,dx$$

$$\lim_{b\to\infty} \int_2^{b} \frac{1}{\cancel{x}\sqrt{u}} \cdot \cancel{x}\,du$$

$$\lim_{b\to\infty} \int_2^{b} u^{-1/2}\,du$$

Well Done!

$$\lim_{b\to\infty} 2\sqrt{\ln x}\,\Big|_2^{b}$$

$$\lim_{b\to\infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} \to \infty$$

Since the $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ diverges, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ must also diverge.

6. Find the radius of convergence of the Maclaurin series for $f(x) = e^x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So to find the radius of convergence, let's use the Rat Test.

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= |x| \cdot \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |x| \cdot 0$$

$$= 0$$

So regardless of the value of x, the Rat. Test says such a series always converges (since it gives a result whose absolute value is less than 1), and thus the radius of convergence is infinite.

8. Use a Maclaurin polynomial of at least 7th degree to approximate $\int_0^{0.1} \frac{1}{1+x^3}\,dx$ to 4 decimal places.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3$$

$$\frac{1}{1-(-x^3)} = 1 - x^3 + (-x^3)^2 + (-x^3)^3$$

$$\int_0^{.1} \frac{1}{1+x^3} = \int_0^{.1} 1 - x^3 + x^6 - x^9$$

$$\int_0^{.1} \frac{1}{1+x^3} = \left[x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} \right]_0^{.1}$$

$$\left[\left(.1 - \frac{(.1)^4}{4} + \frac{(.1)^7}{7} - \frac{(.1)^{10}}{10} \right) - (0) \right] = .1000$$

rounded to 4 decimal places

chopped at 4 decimal places

$= .0999$

Excellent!

Yes include both endpoints

10. The radius of convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{3^n (x-4)^n}{n^2}$ is 1/3. Are the endpoints included?

Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}3^{n+1}(x-4)^{n+1}}{(n+1)^2}}{\frac{(-1)^n 3^n (x-4)^n}{n^2}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{n+1}(x-4)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{\cancel{(-1)^n}\cancel{3^n}\cancel{(x-4)^n}}\right|$$

$$\lim_{n\to\infty}|x-4|\cdot\frac{(3n^2)\frac{1}{n^2}}{(n^2+2n+1)\frac{1}{n^2}}$$

$$\lim_{n\to\infty}|x-4|\cdot\frac{3}{1+\frac{2}{n}^{\,0}+\frac{1}{n^2}^{\,0}}$$

$$3|x-4|<1$$

$$|x-4|<\frac{1}{3}$$

$$-\frac{1}{3}<x-4<\frac{1}{3}$$

$$\frac{11}{3}\le x\le\frac{13}{3}$$

For $x=\frac{11}{3}$ → $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n(\frac{11}{3}-4)^n}{n^2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n 3^n(-\frac{1}{3})^n}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1^n}{n^2}$$ p-series w/ $p>1$ $\boxed{\text{converges}}$

Wonderful!

For $x=\frac{13}{3}$ → $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n(\frac{13}{3}-4)^n}{n^2}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$ Try A.S.T.

✓ signs

✓ $\lim_{n\to\infty}\frac{1}{n^2}\to 0$

converges by A.S.T.

✓ $f(x)=\frac{1}{x^2}$

$f'(x)=\frac{-2}{x^3}$ decreasing

Extra Credit (5 points possible): Prove that if the series $\sum^{\infty} a_n$ is absolutely convergent, then the