Exam 1b Calc 3 9/27/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(*x*, *y*) with respect to *y*.

Definition of the partial derivative of eff with respect to wy:

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Good

2. Find an equation for the plane tangent to $z = x^2y + y$ at the point (3,–2).

$$z_0 = 3^2 \cdot (-2) - 2$$
$$= 9 \times (-2) - 2$$
$$= -20$$

$z_x = 2xy$ $\quad z_x(3,-2) = 2 \times 3 \times (-2) = -12$

$z_y = x^2 + 1$ $\quad z_y(3,-2) = 9 + 1 = 10$

$$z - z_0 = z_x(x - x_0) + z_y(y - y_0)$$

$$z + 20 = -12(x - 3) + 10(y + 2)$$

$$z + 20 = -12x + 36 + 10y + 20$$

Excellent

or, $z = -12x + 10y + 36$

3. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy - y^2}{x^2 + y^2}$ does not exist.

Approaching along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0-y^2}{0+y^2} = \boxed{-1}$$

Approaching along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{0-0}{x^2+0} = \boxed{0}$$

Since the limits do not match, the limit does not Exist

Good

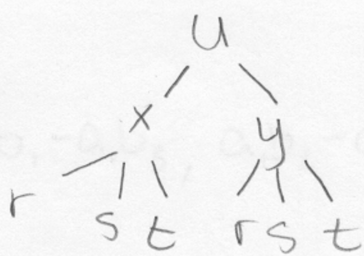
But since I love limits so much...

Approaching along $x=y$

$$\lim_{(x,x)\to(0,0)} \frac{x^2-x^2}{2x^2} = \boxed{0}$$

4. Write the appropriate version of the chain rule for $\dfrac{\partial u}{\partial t}$ in the case where $u = f(x, y)$, $x = x(r, s, t)$, and $y = y(r, s, t)$. Make clear distinction between derivatives and partial derivatives.

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}}$$



All partial derivatives.

Great

5. Let $f(x,y) = y \ln x$. Find the maximum rate of change of f at the point $(2,4)$ and the direction in which it occurs.

rate of change is given by the gradient of a function

$f_x(x,y) = \frac{y}{x}$ $\quad$ $f_y(x,y) = \ln x$

$f_x(2,4) = 2$ $\quad$ $f_y(2,4) = \ln 2$

$\therefore$ gradient $\nabla f = \langle f_x, f_y \rangle$

$= \langle 2, \ln 2 \rangle$

max rate of change $= |\nabla f|$

$= \sqrt{2^2 + (\ln 2)^2}$

$= \sqrt{4 + (\ln 2)^2}$

Excellent

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

To be perpendicular, the dot product must equal zero

So...

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle \cdot \vec{a}$$

$$= \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} + \cancel{a_2 a_3 b_1} - \cancel{a_1 a_2 b_3} + \cancel{a_1 a_3 b_2} - \cancel{a_2 a_3 b_1}$$

$= 0$ since everything cancels!! :)

Nice!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. There was this problem on our exam, and I did everything right except it was like a trick question, because the prof asked for the rate of change in this one direction, right? And I knew that meant a directional derivative, and I like them 'cause it's just a formula, right? But it was unfair because the direction he gave us wasn't a unit vector, and I forgot to make it one. So since it's multiple choice, I got zero credit, even when it was something stupid like that. Why would it have to be a unit vector anyway? I mean, it's just s'posed to be a direction, right, and the direction's the same."

Explain clearly to Biff why it matters to use a unit vector in finding directional derivatives.

Well, to start directional derivative gives the rate of change in any direction which is in dot product with the direction of fastest increase (gradient). Each unique unit vector points in unique direction which can represent the direction in the formula ~~for the~~ of direction vector to find the rate of change in that direction. If we use ~~de~~ other vector in place of unit vector then all it does it gets the gradient magnified.

Here,

$$\nabla f \cdot \hat{u} = |\nabla f||\vec{u}| \cos\theta.$$

So it means that the ~~proo~~ result is the projection of rate of fastest chang in the direction we need ~~to~~ the rate of change. Since $|\vec{u}| = 1$

$$= |\nabla f| \cos\theta.$$

But suppose if ~~we~~ use other vector $\vec{a}$, then, that vector doesnot have $|\vec{a}| = 1$ so

$$\nabla f |\vec{a}| \cdot \cos\theta.$$

all it does ~~is~~ makes the gradient appear bigger and so the rate of change.

Wonderful!

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Explain clearly to Biff why it matters to use a unit vector in finding directional derivatives.

a derivative is a change in a value per change in another variable. A directional derivative is no different. We must have the unit vector magnitude equal to 1 to have a change on 1 unit of magnitude. If we use a vector longer (or shorter) than a unit vector, we do not get a change per every 1 change in another variable. The direction of change will be correct, but the amount of change in that direction at that particular point will not be correct, because what you found was a change over multiple units, not just change per unit.

Wonderful!

8. Find the maximum and minimum values, in the form (x, y, z), of the function $f(x, y) = x^2y$ subject to the constraint $x^2 + y^2 = 36$.

$\nabla f = \lambda \nabla g$

$\nabla f = \langle 2xy, x^2 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\langle 2xy, x^2 \rangle = \lambda \langle 2x, 2y \rangle$

so,

$2xy = \lambda 2x$ ... (i) either, $x = 0$ or, $\lambda = y$

$x^2 = \lambda 2y$ ... (ii)

& $x^2 + y^2 = 36$ ... (iii)

Outstanding!

when, $x = 0$

$y = \pm 6$

the points would be $(0, -6, 0)$ — 1

$(0, 6, 0)$ — 2

If $\lambda = y$ putting the value in (ii)

$x^2 = 2y^2$

putting $x^2 = 2y^2$ in (iii)

$2y^2 + y^2 = 36$

or $3y^2 = 36$

or $y^2 = 12$

Or, $y = \pm\sqrt{12}$

when $y = +\sqrt{12}$

$x^2 + 12 = 36$

$x^2 = 24$

$x = \pm\sqrt{24}$

So, $(-\sqrt{24}, \sqrt{12}, 48\sqrt{3})$ — 3

$(+\sqrt{24}, \sqrt{12}, 48\sqrt{3})$ — 4

when, $y = -\sqrt{12}$ -

$(-\sqrt{24}, -\sqrt{12}, -48\sqrt{3})$ — 5

$(+\sqrt{24}, -\sqrt{12}, -48\sqrt{3})$ — 6

Now,

$f(0, -6) = 0$

$f(0, 6) = 0$

$f(-\sqrt{24}, \sqrt{12}) = 48\sqrt{3}$

$f(+\sqrt{24}, \sqrt{12}) = 48\sqrt{3}$

$f(-\sqrt{24}, -\sqrt{12}) = -48\sqrt{3}$

$f(+\sqrt{24}, -\sqrt{12}) = -48\sqrt{3}$

So, Max^m values are

$(-\sqrt{24}, \sqrt{12}, 48\sqrt{3})$

&

$(+\sqrt{24}, \sqrt{12}, 48\sqrt{3})$

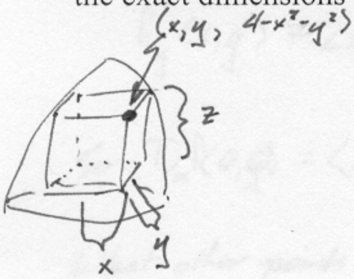
min^m values are.

$(-\sqrt{24}, -\sqrt{12}, -48\sqrt{3})$

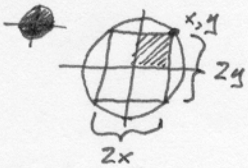
$(+\sqrt{24}, -\sqrt{12}, -48\sqrt{3})$

Ans,

9. You're in charge of a space mission to bring back soil samples from Mars, which may or may not bring contaminants which will destroy all life on Earth. Part of your assignment is to find the largest (in volume) rectangular box that can be fit into the parabolic nosecone of the return module which will bring the samples back to Earth. The nosecone is shaped like the region between $z = 4 - x^2 - y^2$ and the plane $z = 0$ (where the units are in meters). What are the exact dimensions and volume of the largest box possible?



Symmetry - Top View



$$\text{Volume}(x,y) = 4 \cdot x \cdot y \cdot (4 - x^2 - y^2)$$

$$= 16xy - 4x^3y - 4xy^3$$

$$V_x(x,y) = 16y - 12x^2y - 4y^3$$

$$V_y(x,y) = 16x - 4x^3 - 12xy^2$$

$$0 = 4y(4 - 3x^2 - y^2)$$

$$0 = 4x(4 - x^2 - 3y^2)$$

$y = 0$ or $4 - 3x^2 - y^2 = 0$

$\Downarrow$

$(0,0)$ or $4 - x^2 = 0$

(min.)

$\Downarrow$

$x = \pm 2$

$\left.\begin{matrix}(2,0)\\(-2,0)\end{matrix}\right\}$ mins.

$x = 0$ or $4 - x^2 - 3y^2 = 0$

$\Downarrow$

$(0,0)$ or $4 - y^2 = 0$

$\Downarrow$

$y = \pm 2$

$\left.\begin{matrix}(0,2)\\(0,-2)\end{matrix}\right\}$ mins.

$\Downarrow$

$x^2 = 4 - 3y^2$

$\Downarrow$

$4 - 3(4 - 3y^2) - y^2 = 0$

$4 - 12 + 9y^2 - y^2 = 0$

$-8 + 8y^2 = 0$

$8(y^2 - 1) = 0$

$\left.\begin{matrix}y = \pm 1\\x = \pm 1\end{matrix}\right\}$ max!

Dimensions

$2 \times 2 \times 2\ m^3$,

Volume

$8\ m^3$

10. Consider the surface $f(x, y) = x^2 + 2xy + y^2 + 2x$. There is a collection of points all having the same directional derivatives (in any given direction) as the origin. Describe this collection.

$\nabla f(x,y) = \langle 2x + 2y + 2,\ 2x + 2y \rangle$ for (x_0, y_0) any input in the domain of f

$\nabla f(x_0, y_0) = \langle 2x_0 + 2y_0 + 2,\ 2x_0 + 2y_0 \rangle$

The origin's directional derivative is

$\nabla f(0,0) \cdot \vec{u}$ where $\vec{u}$ is a unit vector

$\Rightarrow \langle 2, 0 \rangle \cdot \vec{u}$

So where does $\nabla f(x_0, y_0) = \langle 2, 0 \rangle$

$2x_0 + 2y_0 + 2 = 2$

$2x_0 + 2y_0 = 0 \Rightarrow x_0 = -y_0$

Nice!

$2(-y_0) + 2y_0 + 2 = 0$

$-2y_0 + 2y_0 + 2 = 2$

$2 = 2$

So all points ~~where~~ that satisfy inputs of $x_0 = -y_0$ will have the same directional derivatives in any given direction as the origin