Exam 2 Calc 3 10/26/2007

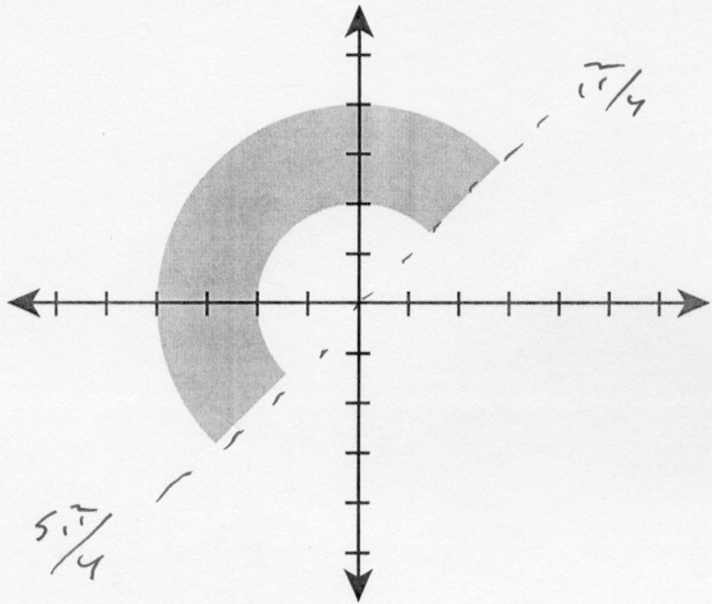
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up limits of integration in polar coordinates for the integral of a function g on the region R shown below:

$$\int_{\theta=\pi/4}^{\theta=5\pi/4}\int_{r=2}^{r=4} g(r,\theta)\, r\, dr\, d\theta$$

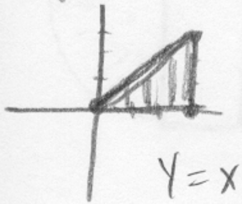
π

Good



2. Set up limits of integration for a double integral to compute $\iint_R f(x,y)\,dA$, where R is the triangle with vertices (0,0), (3,0), and (3,3).

topview



$$\int_0^3 \int_0^x f(x,y)\,dy\,dx$$

$0 \le x \le 3$

$0 \le y \le x$

Great

3. Find the Jacobian for the transformation $x = u^2 + v^2$, $y = u - v$.

Jacobian $= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 1 \\ 2v & -1 \end{vmatrix} = (2u)(-1) - (2v)(1)$

$= -2u - 2v \; du\,dv$

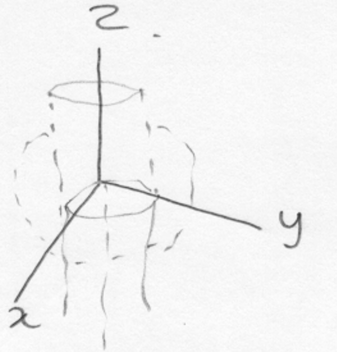
Great

4. Set up an iterated integral for the volume of a sphere (centered at the origin) of radius 3 with a cylinder of radius 2 (centered along the z- axis) removed.

$$x^2+y^2+z^2=9$$

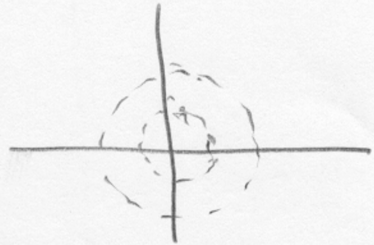
$$z=\sqrt{9-x^2-y^2}$$

$$z=\sqrt{9-r^2}$$



$$\iiint 1 \; dV$$

TOP view

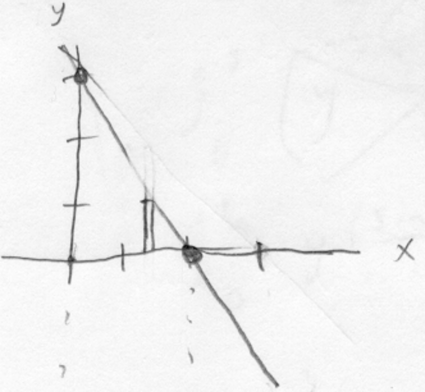


$$2\int_0^{2\pi}\int_2^3\int_0^{z=\sqrt{9-r^2}} 1 \; r\,dr\,d\theta$$

$$2\int_0^{2\pi}\int_2^3\int_0^{z=\sqrt{9-r^2}} r\,dr\,d\theta$$

Nice!

5. Set up an iterated integral for the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.



$X=0 \quad X=2$

$z=0$

$3x+2y=6$

$y = \frac{6-3x}{2}$

$z = 6-3x-2y$

$z_x = -3$

$z_y = -2$

$$\text{Surface Area} = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=\frac{6-3x}{2}} \sqrt{9+4+1}\, dy\, dx$$

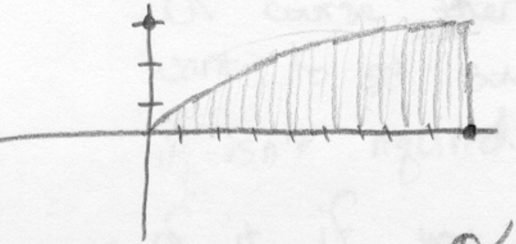
$$= \int_0^2 \int_0^{\frac{6-3x}{2}} \sqrt{14}\, dy\, dx$$

Great

6. Evaluate $\int_0^3 \int_{y^2}^9 y\cos\left(x^2\right) dx\, dy$.

$y\cos(x^2)$

$0 \le y \le 3 \qquad y^2 \le x \le 9$



$dydx$

$0 \le x \le 9$

$0 \le y \le \sqrt{x}$

$$\int_0^9 \int_0^{\sqrt{x}} y\cos(x^2)\, dydx$$

Well done!

$$\int_0^9 \frac{1}{2} y^2 \cos(x^2) \Big]_0^{\sqrt{x}} dx$$

$u = x^2$

$du = 2x$

$\frac{1}{2} du = x$

10

$$\frac{1}{2}\int_0^9 x\cos(x^2)\, dx$$

$$\frac{1}{2}\int_0^9 \frac{1}{2}\cos(u)\, du$$

$$\frac{1}{4}\int_0^9 \cos(u)\, du$$

$\frac{1}{4}\sin 81 - \frac{1}{4}\sin 0$

$$\frac{1}{4}\left(\sin(x^2)\right)\Big|_0^9$$

$$\boxed{\frac{1}{4}\sin(81)}$$

7. Bunny is a calculus student from Enormous State University, and she has a question. Bunny says "So, I really studied hard this chapter, and it's amazing how much more interesting math is when you actually understand it! But there's one thing I was wondering about with, like, changing to polar and stuff. I know there are times when it's lots easier to set something up in x and y, right, and times when it's lots easier in polar. But are there times when you'd really *have* to use x and y, like where polar wouldn't work no matter what, or is it just about what's easier?"

Answer Bunny's question.

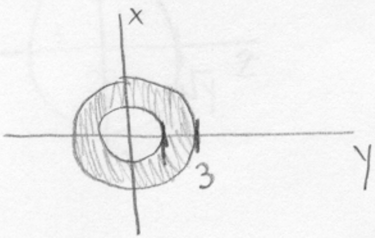
The answer to the second part of your question, Bunny, is yes, what coordinate systems to use in an integration is essentially about what is easier to use. There arent really any situations, however, when polar can't be used, though there are times when converting to polar would be stupid (finding the area of a box, for instance is a lot easier measuring in x and y compared to r and θ.) Using the equations $x = r\cos\theta$ and $y = r\sin\theta$, you'll find that its always possible to convert into polar coordinates.

Note, if you convert to a coordinate system that doesn't work well for your problem, a computer algebra program will make the integration easier.

Fine answer!

8. Set up integrals for the z coordinate of the center of mass of the portion above the xy-plane of the region between a sphere with radius 1 and a sphere with radius 3 (both centered at the origin).

Top View



Above the xy- plane.

This means our ϕ limits are $0 \le \phi \le \pi/2$.

sphere $\Rightarrow$ so integrals for θ are $0 \le \theta \le 2\pi$

2 radii, so the limits are $1 \le \rho \le 3$

So,

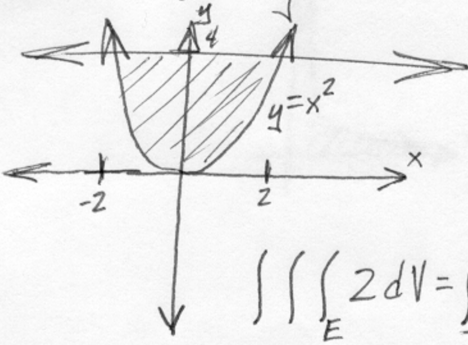
$$\bar{z} = \frac{\int_0^{\pi/2}\int_0^{2\pi}\int_1^3 \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{2\pi}\int_1^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

Excellent!

$$= \boxed{\bar{z} = \frac{\int_0^{\pi/2}\int_0^{2\pi}\int_1^3 \rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{2\pi}\int_1^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi}}$$

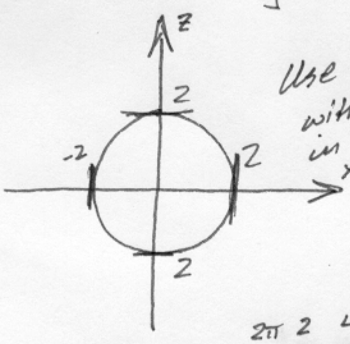
9. Evaluate $\iiint_E 2\,dV$, where E is the region bounded between $y = x^2 + z^2$ and $y = 4$.

Hard Way:



$$\iiint_E 2\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} 2\,dz\,dy\,dx$$

Easier Way:



Use Cylindrical with r and θ in the xz-plane

$$\iiint_E 2\,dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} 2\cdot r\,dy\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 2yr\Big|_{r^2}^{4}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (8r - 2r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[4r^2 - \frac{2}{4}r^4\right]_0^2 d\theta$$

$$= \int_0^{2\pi}(16-8)\,d\theta$$

$$= 8\theta\Big|_0^{2\pi}$$

$$= 16\pi$$

Easiest Way:

$$\iiint_E 2\,dV = 2\iiint_E 1\,dV$$

$$= 2\cdot \text{Volume}_{\text{Paraboloid}}$$

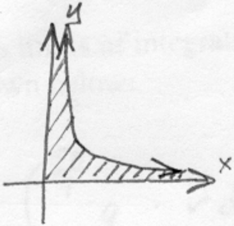
$$= 2\cdot \frac{1}{2}\pi r^2 h$$

$$= \pi\cdot(2)^2\cdot(4)$$

$$= 16\pi$$

10. Let a be some constant. What can you say about the volume of the region in the first octant above $z = a$ but below $z = \dfrac{1}{\sqrt[3]{xy}}$?

Intersection: $a = \dfrac{1}{\sqrt[3]{xy}} \Rightarrow a^3 = \dfrac{1}{xy} \Rightarrow y = \dfrac{1}{a^3 x}$



$$V = \int_0^\infty \int_0^{a^{-3}x^{-1}} \int_a^{x^{-1/3}y^{-1/3}} 1 \, dz\, dy\, dx$$

$$\overset{*}{=} \int_0^\infty \int_0^{a^{-3}x^{-1}} \left(x^{-1/3} y^{-1/3} - a\right) dy\, dx$$

$$\overset{*}{=} \int_0^\infty \left[x^{-1/3} \cdot \frac{3}{2} y^{2/3} - ay \right]_0^{a^{-3}x^{-1}} dx$$

$$= \int_0^\infty \left(\frac{3}{2} x^{-1/3} \cdot a^{-2} \cdot x^{-2/3} - a^{-2} x^{-1} \right) dx$$

$$= \frac{1}{2a} \int_0^\infty x^{-1}\, dx$$

Which we know diverges from Calc 2.

*As long as the resulting limits converge, since these are all improper.