Exam 3a Calculus 3 11/20/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}(x,y,z) = y\,\mathbf{i} + (x-z)\,\mathbf{j} + (2-y)\,\mathbf{k}$ for a line segment C from $(0,3,-2)$ to $(2,0,1)$.

$\langle y, x-z, 2-y \rangle$

pot fun. $yx - zy + 2z$

So by fun. Theorem of line integrals

$$\left[yx - zy + 2z\right]_{(0,3,-2)}^{(2,0,1)} \Rightarrow 0(2) - (1)(0) + 2(1) - \left[3(0) - (-2)(3) + 2(-2)\right]$$

$$2 - [6 - 4]$$

$$2 - 2$$

$$\boxed{0}$$

Good.

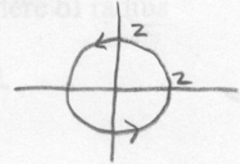
2. Let $\mathbf{F}(x,y,z) = -y\,\mathbf{i} + x\,\mathbf{j} + a\,z\,\mathbf{k}$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is a counterclockwise (when viewed from above) circle of radius 2 centered on the z-axis in the plane $z = 5$.

$0 \le t \le 2\pi$

$x = 2\cos t \qquad \vec{r}(t) = \langle 2\cos t, 2\sin t, 5\rangle$

$y = 2\sin t \qquad \vec{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$z = 5 \qquad \vec{F}(\vec{r}(t)) = \langle -2\sin t, 2\cos t, 5a\rangle$



$$\int_0^{2\pi} \underline{\langle -2\sin t, 2\cos t,} 5a\rangle \cdot \underline{\langle -2\sin t, 2\cos t,} 0\rangle \, dt$$

$$\int_0^{2\pi} 4\sin^2 t + 4\cos^2 t \quad dt$$

$$\int_0^{2\pi} 4(\sin^2 t + \cos^2 t)\, dt \qquad \sin^2 t + \cos^2 t = 1$$

$$\int_0^{2\pi} 4\, dt = \underline{8\pi}$$

Good

3. Evaluate $\int_C \overset{P}{x^4}\,dx + \overset{Q}{xy}\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).

So... Greene's Theorem then

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\iint y - 0 \;dydx$$

$$\int_0^1 \int_0^{1-y} y \;dx\,dy$$

$$\int_0^1 (1-y)y\;dy$$

$$\int_0^1 y - y^2\; dy$$

$$\left.\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$$

$$\frac{1}{2} - \frac{1}{3}$$

$$\frac{1}{6}$$

$(0,1) \rightarrow (1,0)$

$$\frac{1-0}{0-1} = -1$$

$y = -x + b$

$y = -x + 1$

$x = 1 - y$

Great

4. A really goofy cult leader has predicted that at midnight on Thanksgiving, every point within and around the Earth will suddenly begin radiating turkey-flavored jelly beans according to the vector field $\mathbf{F}(x,y,z) = 11x\,\mathbf{i} + 22y\,\mathbf{j} + 2007z\,\mathbf{k}$. Just in case this incredibly unlikely event should come to pass, compute the total number of jelly beans that would radiate outward through the surface of the Earth, i.e. the flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is a sphere of radius approximately 4000 miles.

Closed surface → Divergence Theorem

$$\iiint_E (11 + 22 + 2007)\, dV$$

$$\iiint_E 2040\, dV$$

Volume of a sphere with a radius 4000 multiplied by 2040

$$\frac{4}{3}\pi (4000)^3 \cdot 2040 = 1.7408 \times 10^{14}\, \pi \text{ jelly beans}$$

Nice

5. Compute the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = y\,\mathbf{i} - x\,\mathbf{j} - 2\,\mathbf{k}$ and S is the surface of the paraboloid $z = x^2 + y^2$ (with upward orientation) within the cylinder $x^2 + y^2 = 4$.

$x = u$

$y = v$

$z = u^2 + v^2$

$\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle v, -u, -2 \rangle$

$\vec{r}_u = \langle 1, 0, 2u \rangle$

$\vec{r}_v = \langle 0, 1, 2v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \langle 0-2u, 0-2v, 1-0 \rangle$$

$\langle -2u, -2v, 1 \rangle$

$\iint_D \langle v, -u, -2 \rangle \cdot \langle -2u, -2v, 1 \rangle \, dA$

$\iint_D \cancel{-2uv} + \cancel{2uv} - 2 \, dA$

$\iint_D -2 \, dA$

$\int_0^{2\pi} \int_0^2 -2r \, dr\, d\theta$

$\int_0^{2\pi} -r^2 \Big|_0^2 \, d\theta$

$\int_0^{2\pi} -4 \, d\theta = -8\pi$

Excellent!

6. Show that for any function f(x,y,z) with continuous second partials, curl(grad f) = **0**. Make clear how you use the continuity condition.

well, $\nabla f = \langle f_x, f_y, f_z \rangle$

so, $\text{curl } \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$

$\text{curl } \nabla f = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$

Since the 2nd partials are continuous,

$f_{zy} = f_{yz}$

and

$f_{xz} = f_{zx}$

and

$f_{yx} = f_{xy}$

Excellent!

Since 2nd partials are equal,

$f_{zy} - f_{yz} = 0$

$f_{xz} - f_{zx} = 0$

$f_{yx} - f_{xy} = 0$

and thus, $\langle 0, 0, 0 \rangle = \boxed{\vec{0}}$

7. Buffy says "Like, Calculus is *so* unfair. My professor gave us this test, and we were supposed to show that for this line integral thingy it didn't matter what way you went, like the path or whatever. So I like, did it for two different parama-whatevers, and, like, I got the same thing both ways, which is like a miracle for me anyway, but my professor wrote all this bad stuff that I totally don't understand about how that wasn't the right thing to do. She gave me almost no credit at all! But, my *God*, how else could I do it? I mean, it's not like I have enough time to do every way of connecting those two points, cause there's, like, *lots* of them."

Explain (clearly enough for Buffy to understand) how such a thing can be done without actually trying an infinite number of paths, and tell her what's flawed about her approach.

Well, the fundamental idea that you're missing is that the path may be irrelevant to what the solutions for your line integrals are. In this case, in a conservative vector field, only where your path starts and ends determine your final answer. You can tell if a vector field is conservative in two ways.

Excellent!

If a vector field has the form $\langle P, Q \rangle$, then check to see in Q_x and P_y are equal.

If a vector field has the form $\langle P, Q, R \rangle$, check to see if the curl of the vector field is $\vec{0}$. (This is $\nabla \times \vec{F}$)

In both cases, you are determining whether or not your vector field is the gradient of another function called the potential function (f)

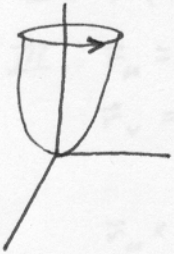
If a vector field does have a potential function (f), then you can use the Fun. Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})$$

8. Evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = 2x^2\,\mathbf{i} + \mathbf{j} + xy\,\mathbf{k}$ and S is the surface of the paraboloid $z = x^2 + y^2$ (with upward orientation) within the cylinder $x^2 + y^2 = 4$.

Stokes' Theorem!

Instead of doing the surface integral directly, we'll use Stoke's Theorem to convert it to a line integral around the surface's boundary, which is a circle of radius 2 in the plane $z=4$:



I. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$

II. $\vec{F}(\vec{r}(t)) = \langle 2(2\cos t)^2, 1, 2\cos t \cdot 2\sin t\rangle$

III. $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

IV. $\int \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 8\cos^2 t, 1, 4\sin t\cos t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\,dt$

$$= \int_0^{2\pi} (-16\cos^2 t\cdot\sin t + 2\cos t + 0)\,dt$$

let $u = \cos t$

$\frac{du}{dt} = -\sin t$

$\frac{du}{-\sin t} = dt$

$$= -16\int_0^{2\pi} \cos^2 t\cdot\sin t\,dt + \underbrace{2\int_0^{2\pi}\cos t\,dt}_{\text{zero by symmetry}}$$

$$= -16\int_{t=0}^{t=2\pi} u^2\cdot\cancel{\sin t}\cdot\frac{du}{-\cancel{\sin t}} + 0$$

$$= -16\cdot -1\cdot \frac{u^3}{3}\Big|_{t=0}^{t=2\pi}$$

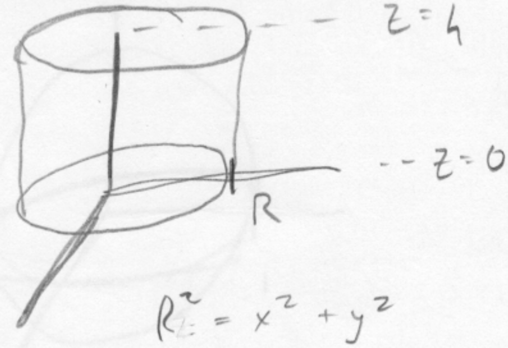
$$= 16\cdot\frac{1}{3}\cos^3 t\,\Big|_0^{2\pi}$$

$$= \frac{16}{3}\cdot(\cos^3 2\pi - \cos^3 0)$$

$$= \frac{16}{3}\cdot(1-1)$$

$$= 0$$

9. Let $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and let S be the portion of a cylinder of radius R centered on the z-axis between $z = 0$ and $z = h$ for some positive constant h, with outward orientation. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.



I Parametrize

$0 \le u \le 2\pi$

$0 \le v \le h$

$x = R\cos u$

$y = R\sin u$

$z = v$

$\vec{r}(u,v) = \langle R\cos u, R\sin u, v\rangle$

II Build $\vec{F}(r(u,v)) = \langle R\cos u, R\sin u, v\rangle$

III $\vec{r}_u = \langle -R\sin u, R\cos u, 0\rangle$

$\vec{r}_v = \langle 0, 0, 1\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin u & R\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle R\cos u, -(-R\sin u), 0\rangle$$

IV & V Set up integral & solve

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle R\cos u, R\sin u, v\rangle \cdot \langle R\cos u, R\sin u, 0\rangle \, dA$$

$$= \int_{u=0}^{u=2\pi} \int_{v=0}^{v=h} \underbrace{R^2\cos^2 u + R^2\sin^2 u}_{R^2(\cos^2 u + \sin^2 u) = R^2} \, dv\, du$$

$$= R^2 \int_0^{2\pi} \int_0^h 1 \, dv\, du \qquad [v]_0^h$$

$$= R^2 \int_0^{2\pi} h\, du \qquad [hu]_0^{2\pi}$$

$$= R^2$$

$$= \underline{2\pi h R^2}$$

Well done!

10. Show that the surface area of a unit sphere (with parametrization $x(u,v) = \sin u \cos v$, $y(u,v) = \sin u \sin v$, $z(u,v) = \cos u$) is 4π.

$\iint |r_u \times r_v| \, dA$

$r(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

$0 \leq u \leq \pi \qquad 0 \leq v \leq 2\pi$

$r_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$

$r_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v, & \sin u \cos v & 0 \end{vmatrix} = \langle 0 + \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \cos^2 v + \sin u \cos u \sin^2 v \rangle$$

$\iint \sqrt{(\sin^2 u \cos v)^2 + (\sin^2 u \sin v)^2 + (\sin u \cos u)^2}$

$\iint \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u}$

$\iint \sqrt{\sin^4 u + \sin^2 u \cos^2 u}$

$\iint \sin u \sqrt{\cancel{\sin^2 + \cos^2}}$

Nice.

$\int_0^{2\pi} \int_0^{\pi} \sin u \, du \, dv = \int -\cos u \Big|_0^{\pi} dv = \int 1 - -1 \, dv$

$\int_0^{2\pi} 2 \, dv = \boxed{4\pi}$