Exam 1a   Calc 3   9/30/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(x, y) with respect to x.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Good

2. Find an equation for the plane tangent to $z = 3x^2 + 2y^2 - 2y - 1$ at the point (3,5,66).

$$z = 3x^2 + 2y^2 - 2y - 1 \qquad (3,5,66)$$

$$z_x = 6x \qquad\qquad z_y = 4y - 2$$

$$z_{(3,5,66)} = 18 \qquad\qquad z_{(3,5,66)} = 20 - 2 = 18$$

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

$$z - 66 = 18(x - 3) + 18(y - 5)$$

$$z = 18x - 54 + 18y - 90 + 66$$

$$z = 18x + 18y - 78$$

Great!

3. Show that $\lim_{(x,y)\to(0,0)} \frac{xy - y^2}{x^2 + y^2}$ does not exist.

- Approach $(0,0)$ along $x = 0$:

$$(-)\quad \lim_{(0,y)\to(0,0)} \frac{xy - y^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0.y - y^2}{0^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

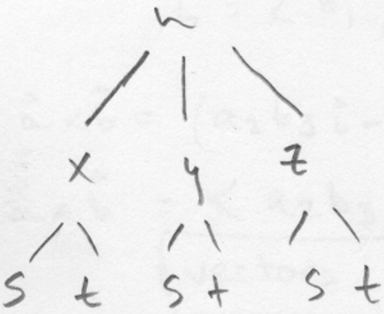
- Approach $(0,0)$ along $y = 0$

$$(-)\quad \lim_{(x,0)\to(0,0)} \frac{xy - y^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x.0 - 0^2}{x^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

Since the limits are different as we approach different paths, the limit does not exist.

Excellent!

4. Write the appropriate version of the chain rule for $\frac{\partial u}{\partial t}$ in the case where $u = f(x, y, z)$, $x = x(s, t)$, $y = y(s, t)$, and $z = z(s, t)$. Make clear distinction between derivatives and partial derivatives.



$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}}$$

*All Partials

Good

5. Find the maximum and minimum values (and distinguish clearly which are which) of the function $f(x,y) = 4x + 6y$ subject to the constraint $x^2 + y^2 = 16$.

Lagrange!

$$\boxed{\begin{array}{c} \nabla f = \lambda \nabla g \\ g = k \end{array}}$$

$$\langle 4, 6 \rangle = \lambda \langle 2x, 2y \rangle$$

$$x^2 + y^2 = 16$$

$$4 = 2x\lambda \Rightarrow \cancel{\lambda = \tfrac{2}{x}} \quad x = \frac{2}{\lambda}$$

$$6 = 2y\lambda \Rightarrow \cancel{\lambda = \tfrac{3}{y}} \quad y = \frac{3}{\lambda}$$

$$\left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 16$$

$$\frac{4}{\lambda^2} + \frac{9}{\lambda^2} = 16$$

$$\frac{13}{\lambda^2} = 16 \Rightarrow \lambda^2 = \frac{13}{16} \Rightarrow \lambda = \pm \frac{\sqrt{13}}{4}$$

For $\lambda = \frac{\sqrt{13}}{4}$:

$$x = \frac{8}{\sqrt{13}}, \quad y = \frac{12}{\sqrt{13}} \quad \text{max.}$$

For $\lambda = \frac{-\sqrt{13}}{4}$:

$$x = \frac{-8}{\sqrt{13}} \quad y = \frac{-12}{\sqrt{13}} \quad \text{min.}$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$. $\langle a\times b\rangle \cdot \vec{b} = 0$

$\vec{a} = \langle a_1, a_2, a_3\rangle$ $\quad$ $\vec{b} = \langle b_1, b_2, b_3\rangle$

$a\times b : a_2b_3\,i + a_3b_1\,j + a_1b_2\,k \; -$

$$\begin{matrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$$

$-\,[a_2b_1\,k + a_3b_2\,i + a_1b_3\,j]$

$$\begin{array}{l} a_2b_3\,i + a_3b_1\,j + a_1b_2\,k \\ -\;a_3b_2\,i + a_1b_3\,j + a_2b_1\,k \\ \hline \end{array}$$

$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle \cdot \langle b_1, b_2, b_3\rangle$

$b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3 + b_3(a_1b_2 - a_2b_1)$

$\boxed{b_1a_2b_3} - b_1a_3b_2 + b_2a_3b_1 - a_1b_3b_2 + b_3a_1b_2 \boxed{- b_3a_2b_1} = 0$

Since two vectors are perpendicular when their dot product is 0, and $\vec{a}\times\vec{b}\cdot\vec{b} = 0$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$.

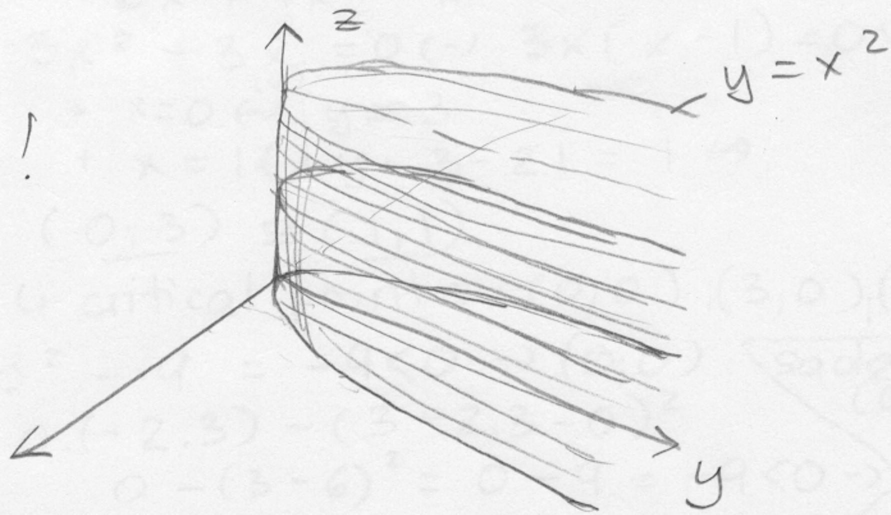
Nice!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. It's like, geez, everything I had to memorize before in math is all a lie now, you know? Like on our first test, there was this question about what the graph of $y = x^2$ looked like, right? And obviously I memorized that before, so I picked the parabola graph, which was right there answer b in the multiple choices, right? But they said that one was wrong because of the z, which is crazy 'cause there's not even a z in the equation."

Explain clearly to Biff why the graph of $y = x^2$ might not look like it used to.

Remember Biff. This is Cal 3! And we deal with 3 dimensions. So look at: $y = x^2$, you see there's no "z". That means "z" can be everything (all values). That would make the graph of $y = x^2$ a form of cylinder, not a parabola. Because, the graph will contain various $y = x^2$ in the x-y plane (which is a parabola) running along the "z" axis. Like this:

Nice!



So you can see for yourself. The graph is not a parabola at all

8. Find and classify all critical points of the function $f(x, y) = 3xy - x^2y - xy^2$.

$f_x = 3y - 2xy - y^2$ $\qquad 0 = y(3 - 2x - y) \Rightarrow y = 0$ or $y = 3 - 2x$

$f_y = 3x - \cancel{y}x^2\cancel{y} - 2xy$ $\qquad 0 = x(3 - x - 2y)$

$\Downarrow$

$x = 0$ or $x = 3$

$\Downarrow$

$x = 0$ or $3 - x - 2(3-2x)$

$-3 + 3x = 0$

$x = 1$

$f_{xx} = -2y \qquad f_{xy} = 3 - 2x - 2y \qquad f_{yy} = -2x$

So we have critical points

$(0,0), (3,0), (0,3), (1,1)$

To Classify:

$D(0,0) = (0)(0) - (3)^2 = -9 < 0$ saddle point

$D(3,0) = (0)(-6) - (-3)^2 = -9 < 0$ saddle point

$D(0,3) = (-6)(0) - (-3)^2 = -9 < 0$ saddle point

$D(1,1) = (-2)(-2) - (-1)^2 = 3 > 0$ and $f_{xx} < 0$ so local max

9. Show that the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as $\dfrac{x x_0}{a^2} + \dfrac{y y_0}{b^2} - \dfrac{z z_0}{c^2} = 1$.

$f_x = \dfrac{2x}{a^2}$ $\quad f_y = \dfrac{2y}{b^2}$ $\quad f_z = \dfrac{-2z}{c^2}$

equation for tangent plane at (x_0, y_0, z_0) can be written as

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$$

$$\frac{2x_0x}{a^2} - \frac{2x_0^2}{a^2} + \frac{2y_0y}{b^2} - \frac{2y_0^2}{b^2} - \frac{2z_0z}{c^2} + \frac{2z_0^2}{c^2} = 0$$

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}$$

Nicely Done!

we know (x_0, y_0, z_0) lies on the hyperbola,

so $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} - \dfrac{z_0^2}{c^2} = 1$

so

$$\boxed{\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{z z_0}{c^2} = 1}$$

10. Jon has discovered an abandoned silver mine on family property in Arizona, and plans to resume mining operations as soon as possible (with the revenues going to a fund for enormous prize scholarships for the students earning the highest scores in Calculus 3 classes). However, it will first be necessary to construct a new access road. The mine is located at the point (4000, 2000, 4000) on Ideal Mountain, which has equation $z = \sqrt{6000^2 - x^2 - y^2}$. The road can have at most a 5% grade, meaning that as a truck moves along the road leaving the mine it cannot descend more that 5 feet for every 100 feet of horizontal travel. In which direction should the road leave the mine to descend as rapidly as feasible?

It must have something to do with the gradient, so

$$\nabla z = \left\langle \tfrac{1}{2}(6000^2 - x^2 - y^2)^{-1/2} \cdot -2x, \tfrac{1}{2}(6000^2 - x^2 - y^2)^{-1/2} \cdot -2y \right\rangle$$

$$= \left\langle \frac{-x}{\sqrt{6000^2 - x^2 - y^2}}, \frac{-y}{\sqrt{6000^2 - x^2 - y^2}} \right\rangle$$

so at the point in question:

$$\nabla z = \left\langle \frac{-4000}{4000}, \frac{-2000}{4000} \right\rangle = \left\langle -1, -\tfrac{1}{2} \right\rangle.$$

So we need a direction $\langle a, b \rangle$ for which the directional derivative is $\frac{-5}{100} = \frac{-1}{20}$, or

$$\nabla z \cdot \langle a, b \rangle = \frac{-1}{20} \Rightarrow \langle -1, -\tfrac{1}{2} \rangle \cdot \langle a, b \rangle = \frac{-1}{20} \Rightarrow -a - \tfrac{1}{2} b = \frac{-1}{20}$$

And $\langle a, b \rangle$ also needs to be a unit vector, so $a^2 + b^2 = 1$, leading to

$$\left(\frac{1}{20} - \frac{1}{2} b\right)^2 + b^2 = 1$$

or

$$\frac{5}{4} b^2 - \frac{1}{20} b - \frac{399}{400} = 0$$

which my TI-89 says has solutions

$$b \approx .9135 \text{ or } b \approx -.8735$$

with corresponding components

$$a \approx -.4068 \text{ or } a \approx .4868$$