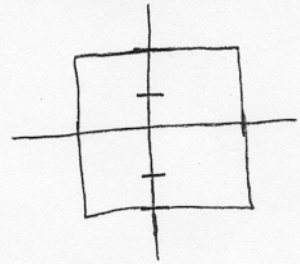
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up a double integral for the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the ~~triangle~~ rectangle $R = [-1, 1] \times [-2, 2]$.

$$z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$$

Great!



$$V = \int_{x=-1}^{x=1} \int_{y=-2}^{y=2} \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dy\, dx$$

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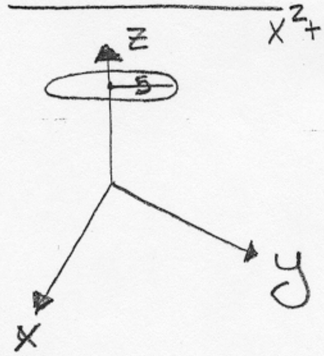
2. Set up a triple integral for the first-octant volume below $z = xy$ inside a cylinder with radius 5 centered on the z-axis.

$x^2 + y^2 = 25$



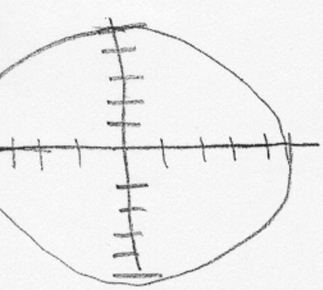
$$V = \int_{x=0}^{x=5} \int_{y=0}^{y=\sqrt{25-x^2}} \int_{z=0}^{z=xy} 1 \, dz\, dy\, dx$$

Excellent!

$x = 0$

$y = 0$

$z = 0$



$y = \pm\sqrt{25 - x^2}$

but 1st octant

so $y = \sqrt{25 - x^2}$

3. Set up an iterated integral for the surface area of the portion of the parabolic cylinder $z = y^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

$$SA = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$Z = y^2$

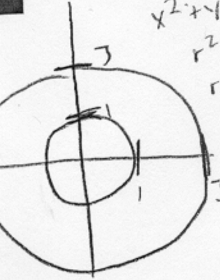
$Z_x = 0$

$Z_y = 2y = 2r\sin\theta$

$x^2 + y^2 = 9$

$r^2 = 9$

$r = 3$



$x^2 + y^2 = 1$

$r^2 = 1$

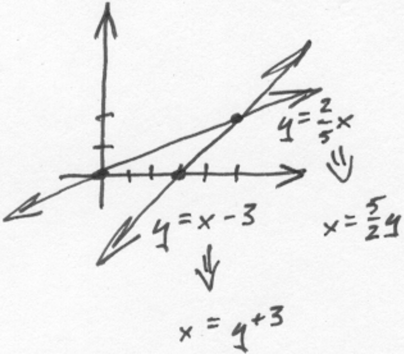
$r = 1$

Good

$$\int_0^{2\pi} \int_1^3 \sqrt{1 + (2r\sin\theta)^2}\; r\, dr\, d\theta$$

$$\int_0^{2\pi} \int_1^3 \sqrt{1 + 4r^2\sin^2\theta}\; r\, dr\, d\theta$$

4. Set up iterated integrals for the x coordinate of the center of mass of the triangular region with vertices (0,0), (3,0), and (5,2), given that its density at each point is proportional to the distance of that point from the y-axis.



$$\rho(x,y) = k \cdot x$$

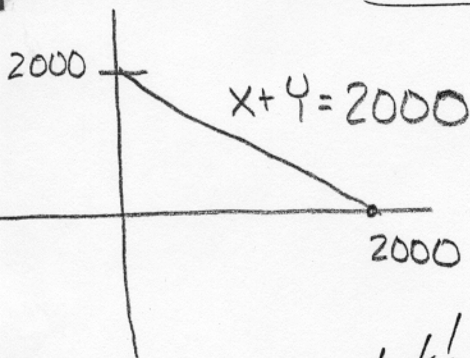
$$\bar{x} = \frac{\displaystyle\int_{y=0}^{y=2}\int_{x=\frac{5}{2}y}^{x=y+3} x \cdot kx \, dx\,dy}{\displaystyle\int_{y=0}^{y=2}\int_{x=\frac{5}{2}y}^{x=y+3} kx \, dx\,dy}$$

5. Suppose that a lamp has a bulb with a mean lifetime $\mu = 1000$ hours, which can be modeled with an exponential density function, and that as soon as one bulb burns out a second bulb replaces it. Set up an iterated integral for the probability that both bulbs burn out within a total of 2000 hours.

General equation

$$P: X + Y = 2000$$

$$P = \begin{cases} 0 & \text{if } t < 0 \\ 1000^{-1} e^{-t/1000} & t \geq 0 \end{cases}$$



$$P = \int_{X=0}^{X=2000} \int_{Y=0}^{Y=2000-X} \underbrace{\left(\frac{1}{1000} e^{-X/1000}\right)}_{1^{st}\text{ bulb}} \underbrace{\left(\frac{1}{1000} e^{-Y/1000}\right)}_{2^{nd}\text{ bulb}} dy\,dx$$

Excellent!

6. Find the Jacobian for the transformation $x = uv$, $y = vw$, $z = uw$.

$$\text{Jacobian} = \begin{vmatrix} v & 0 & w \\ u & w & 0 \\ 0 & v & u \end{vmatrix} = vwu + wuv$$

$$= \underline{\underline{2uvw}}$$

Great

7. Biff is a calculus student from Enormous State University, and he has a question. Biff says "So, these double integrals are killin' me. On our quiz there was this one where I got it wrong and the TA said something about how I did it too much, like I did it for a rectangle, but it was supposed to be for a triangle. So then on the exam, I divided my answers by two, 'cause a triangle is half of a rectangle, right? But then they marked those wrong too. So what's up with that?"

Explain to Biff why his plan to divide by two for triangular regions does or doesn't work.

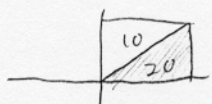
I would begin explaining this to ~~Karl~~ Biff by proposing a very silly function, and jumping back to single integrals. Say we have the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

now suppose we were supposed to find the integral from 0 to 1, but instead you took the integral from -1 to 1, and divided by two.

$\int_0^1 f(x) = 1.$ $\quad \dfrac{\int_{-1}^1 f(x)}{2} = \frac{1}{2}.$

If it doesnt work in one dimension, it would be a little silly for it to work in two.

Now consider this graph 

where the numbers indicate the volume of the regions.

$\int$ shaded $= 20$, $\int$ not shaded $= 10$. $\int$ rectangle $= 30$. $\dfrac{\int \text{rectangle}}{2} = 15$, which is not the volume of either sub-region.

I like it!

8. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} 5\,dz\,dy\,dx$.

That's 5 times the volume of the top half of a sphere with radius 2,

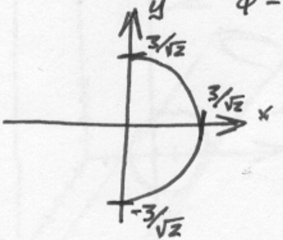
so it's

$$5 \cdot \frac{1}{2} \cdot \frac{4}{3}\pi(2)^3$$

$$= \frac{80\pi}{3}$$

9. Set up iterated integrals for the x coordinate of the center of mass of the region bounded by the cone $z = \sqrt{x^2 + y^2}$, the sphere $x^2 + y^2 + z^2 = 9$, and with $x \geq 0$.

$\phi = \pi/4$ $\qquad \rho = 3 \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

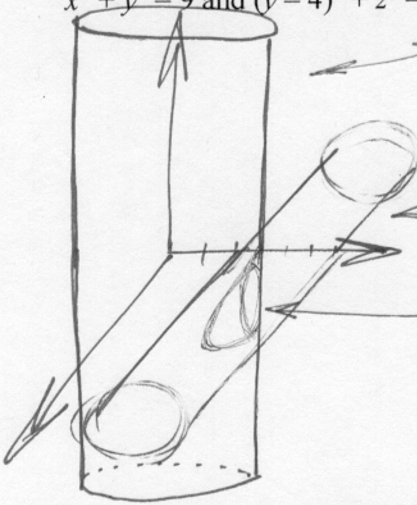


$\rho \sin\phi \cos\theta$

$$\bar{x} = \frac{\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=3} x \cdot k \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}{\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=3} 1 \cdot k \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

$$= \frac{\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=3} \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta}{\int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

10. Set up an iterated integral (or integrals) for the volume of the region bounded between the surfaces $x^2 + y^2 = 9$ and $(y-4)^2 + z^2 = 4$.



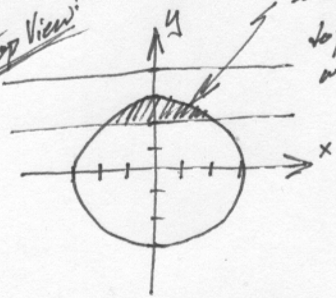
Cylinder centered on z-axis

Cylinder parallel to x-axis

"Notch" lateral cylinder takes out of bigger vertical cylinder

So our region extends from bottom to top of the lateral cylinder, within this top view.

Top View:



Based on the top view, find the x bounds by setting $y=2$ in $x^2+y^2=9$, or

$$x^2 + (2)^2 = 9$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

$$\text{Volume} = \int_{x=-\sqrt{5}}^{x=\sqrt{5}} \int_{y=2}^{y=\sqrt{9-x^2}} \int_{z=-\sqrt{4-(y-4)^2}}^{z=\sqrt{4-(y-4)^2}} 1\,dz\,dy\,dx$$