Exam 3a Calculus 3 11/25/2008

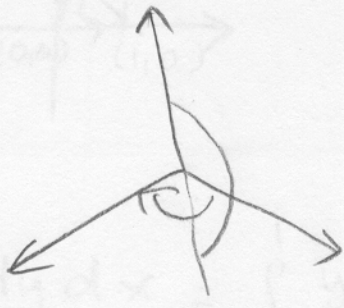
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Parametrize and give limits for a sphere with radius 2.

$x = 2\sin\phi\cos\theta$
$y = 2\sin\phi\sin\theta$
$z = 2\cos\phi$

Good

$0 \leq \phi \leq \pi$
$0 \leq \theta \leq 2\pi$



2. Evaluate $\int_C \mathbf{G}\cdot d\mathbf{r}$, where $\mathbf{G}(x, y) = \langle xy^2, x^2y\rangle$ and C is a line segment from (2,1) to (5,–4).

$\frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x} = 2xy$

$\to$ G is conservative

potential function: $G(x,y) = \frac{1}{2}x^2y^2 + K$

$\to \int_C G\cdot dr = G(5;-4) - G(2;1)$

$= \frac{1}{2}\cdot 5^2(-4)^2 - \frac{1}{2}(2)^2 1^2$

$= \frac{1}{2}\cdot 25\cdot 16 - \frac{1}{2}\cdot 4$

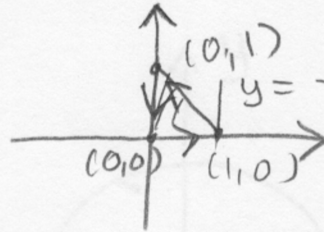
Great

$= 200 - 2 = 198$

3. Evaluate $\int_C x^4 dx + xy\, dy$, where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).

Closed Paths

-> Green's Theorem :



$$\int_C x^4 dx + xy\, dy$$

Great!

$$= \int_0^1 \int_0^{-x+1} y - (0)\, dy\, dx = \int_0^1 \int_0^{-x+1} y\, dy\, dx = \int_0^1 \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 1 - 2x + x^2 dx = \frac{1}{2}\left(x - x^2 + \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(1 - 1 + \frac{1}{3}\right) = \frac{1}{6}$$

4. Set up an integral for the surface area of the helicoid with vector equation $\mathbf{r}(u, v) = u \cos v \,\mathbf{i} + u \sin v \,\mathbf{j} + v \,\mathbf{k}$, for $0 \le u \le 1$, $0 \le v \le \pi$.

Surface Area $= \iint |r_u \times r_v| \, dS$

$$\boxed{\int_0^{\pi}\int_0^{1} \sqrt{1 + u^2} \, du \, dv}$$

$r = \langle u\cos v, u\sin v, v \rangle$

$r_u = \langle \cos v, \sin v, 0 \rangle$

$r_v = \langle -u\sin v, u\cos v, 1 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

Excellent!

$\langle (\sin v - 0), -(\cos v - 0), (u\cos^2 v + u\sin^2 v) \rangle$

$\langle \sin v, -\cos v, u \rangle$

$\sqrt{\sin^2 v + \cos^2 v + u^2}$

5. Set up an integral for $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle x^4, xy - y^2\rangle$ and C is the top half of a circle with radius 2, centered at the origin and traversed counterclockwise.

not closed no Green / not Curl no Stokes / no pot

Long way

$x = 2\cos t \qquad 0 \le t \le \pi$

$y = 2\sin t$

$r(t) = \langle 2\cos t, 2\sin t\rangle$

$r'(t) = \langle -2\sin t, 2\cos t\rangle$

$\vec{F}(r(t)) = \langle 16\cos^4 t, 4\cos t\sin t - 4\sin^2 t\rangle$

$\int_0^{\pi} \langle 16\cos^4 t, 4\cos t\sin t - 4\sin^2 t\rangle \cdot \langle -2\sin t, 2\cos t\rangle$

Great

$\int_0^{\pi} (-32\cos^4 t\sin t + 8\cos^2 t\sin t - 8\sin^2 t\cos t)\,dt$

$8\int_0^{\pi} (\cos^2 t\sin t - \sin^2 t\cos t - 4\cos^4 t\sin t)\,dt$

6. Show that for any function f(x,y,z) with continuous second partials, curl(grad f) = **0**. Make clear how you use the continuity condition.

$$\text{grad } f = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\text{grad } f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} =$$

$$\langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle$$

Since the second order partials are continuous...

$$f_{xy} = f_{yx}$$

$$f_{xz} = f_{zx}$$

Nice

$$f_{yz} = f_{zy}$$

$$\langle 0, 0, 0 \rangle = \vec{0}$$

7. Biff says "Okay, so I got it figured out. Calculus isn't really math, dude. Math is when, like there's numbers and you work out an answer. On our test, though, the last question was about, like, if you know a vector field is conservative, then what can you say about the line integrals on two legs of a right triangle and the line integral on the hypotenuse. That's obviously, like, a philosophy question or something, not a math question, 'cause it's totally asking for an opinion!"

Explain (clearly enough for Biff to understand) what sort of non-philosophical answer might be given to such a question.

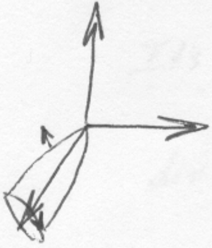
In a conservative vector field, the line integral on any path you take that ends up where it started is 0. A right triangle is one such path: if you trace the outline of a right triangle you end up where you started, regardless of where you started. Naturally

$$\int_{L_1} \vec{F} d\vec{r} + \int_{L_2} \vec{F} d\vec{r} + \int_{\text{Hypoteneuse}} \vec{F} d\vec{r} = 0.$$

So, to avoid philosophy, you could say $\int_H \vec{F} d\vec{r} = -\left[\int_{L_1} \vec{F} d\vec{r} + \int_{L_2} \vec{F} d\vec{r}\right]$, or that the line integral of the hypotoneuse is the negative of the sum of the line integrals of the two legs.

Excellent!

8. Set up an integral for $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle y, z, x\rangle$ and S is the portion of the paraboloid $x = y^2 + z^2$ with $x \le 9$, with outward orientation.



Note orientation, which leads to funny paramaterization.

Let's use Stokes' Theorem, so we do a line integral on the circle that's the boundary of the paraboloid.

$\vec{r}(t) = \langle 9, 3\sin t, 3\cos t\rangle$ for $0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle 3\sin t, 3\cos t, 9\rangle$

$\vec{r}\,'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$$\iint \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \langle 3\sin t, 3\cos t, 9\rangle \cdot \langle 0, 3\cos t, -3\sin t\rangle \, dt$$

$$= \int_0^{2\pi} \left(0 + 9\cos^2 t - 27\sin t\right) dt$$

9. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a sphere centered at the origin with radius 3 and outward orientation and $\mathbf{F}(x, y, z) = \langle xz, -z, y \rangle$.

It's a closed surface, so let's use the Divergence Theorem!

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(y)$$

$$= z + 0 + 0$$

$$= z$$

So $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E z \, dV$$

$$= 0$$ (Since that's the numerator of $\bar{z}$ for a sphere centered at the origin, which turns out to be zero.)

10. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the disc of radius 3 centered at the origin in the xy-plane with downward orientation and $\mathbf{F}(x, y, z) = \langle xz, -z, y \rangle$.

I. $\vec{r}(u,v) = \langle u, v, 0 \rangle$

II. $\vec{F}(\vec{r}(u,v)) = \langle 0, -0, v \rangle$

III. $\vec{r}_u = \langle 1, 0, 0 \rangle$

$\vec{r}_v = \langle 0, 1, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \vec{i} \times \vec{j} = \vec{k} = \langle 0, 0, 1 \rangle$

But we wanted downward orientation, so reverse this to $\langle 0, 0, -1 \rangle$.

IV. $$\iint_S \vec{F} \circ d\vec{S} = \iint_D \langle 0, 0, v \rangle \circ \langle 0, 0, -1 \rangle \, dA$$

$$= \iint_D -v \, dA$$

$$= \int_0^{2\pi} \int_0^3 -r \sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} -\frac{r^3}{3} \sin\theta \Big|_{r=0}^{r=3} d\theta$$

$$= 9\cos\theta \Big|_0^{2\pi}$$

$$= 0$$