Exam 3b Calculus 3 11/25/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Let $\mathbf{F}(x, y, z) = \langle 3x, y^3, z^5 \rangle$. Compute curl $\mathbf{F}$.

Curl $= \nabla \times F$

Excellent!

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x & y^3 & z^5 \end{vmatrix} = \left\langle \frac{\partial(z^5)}{\partial y} - \frac{\partial(y^3)}{\partial z}, -\left(\frac{\partial(z^5)}{\partial x} - \frac{\partial(3x)}{\partial z}\right), \frac{\partial(y^3)}{\partial x} - \frac{\partial(3x)}{\partial y} \right\rangle$$

$$\langle 0-0, 0-0, 0-0 \rangle$$

$$\boxed{\vec{0}}$$

2. Evaluate $\int_C \mathbf{G} \cdot d\mathbf{r}$, where $\mathbf{G}(x, y) = \langle \underbrace{xy^2}_{P}, \underbrace{x^2y}_{Q} \rangle$ and C is a line segment from (2,1) to (5,–4).

$\frac{\partial P}{\partial y} = 2xy$

so $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ By Clairaut's Theorem

$\frac{\partial Q}{\partial x} = 2xy$

Use Fundamental Theorem of line Integrals

$g = \frac{1}{2}x^2y^2$

Good

$\int_C \vec{G} \cdot d\vec{r} = g(5,-4) - g(2,1)$

$\frac{1}{2}(5)^2(-4)^2 - \frac{1}{2}(2)^2(1)^2$

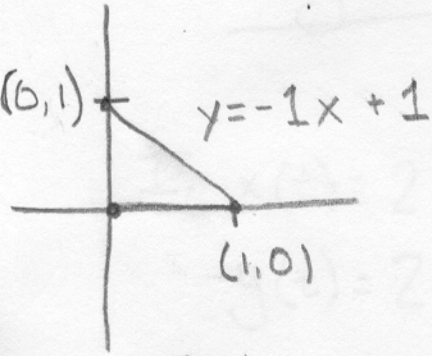
$200 - 2$

$\boxed{198}$

3. Evaluate $\int_C \underbrace{x^4}_{P}\,dx + \underbrace{xy}_{Q}\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).

no potential function but a closed curve

Use Green's Theorem



$$\int_C P\,dx + Q\,dy = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\iint_C (y - 0)\,dA$$

$$\frac{0-1}{1-0} = \frac{-1}{1} = -1$$

$$\int_0^1 \int_0^{y=1-x} y\,dy\,dx$$

Well done

$$\boxed{\frac{1}{6}}$$

W

4. Set up an integral for the surface area of the helicoid with vector equation $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, for $0 \le u \le 1$, $0 \le v \le \pi$.

$$SA = \iint_D |\vec{r}_u \times \vec{r}_v|\,dV$$

Excellent!

D

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle \qquad 0 \le u \le 1, \quad 0 \le v \le \pi$$

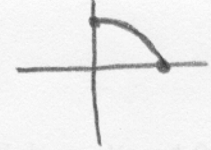
$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$$

$$\boxed{SA = \int_0^{\pi}\int_0^{1} \sqrt{1+u^2}\,du\,dv}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\cos^2 v + u\sin^2 v\rangle = \langle \sin v, -\cos v, u\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(\sin v)^2 + (-\cos v)^2 + u^2} = \sqrt{1+u^2}$$



5. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy^3, 2y^4\rangle$ and C is the first-quadrant arc of a circle with radius 2, traversed counterclockwise.

not closed ⇒ NO Green's

Long Way!

$\frac{\partial P}{\partial y} = 3xy^2$

$\frac{\partial Q}{\partial x} = 0$

⇒ can't use Fun. Theorem of Line Integrals

I. $x(t) = 2\cos t$

$y(t) = 2\sin t$

$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ for $0 \le t \le \frac{\pi}{2}$

II. $\vec{F}(\vec{r}(t)) = \langle 2\cos t(2\sin t)^3, 2(2\sin t)^4\rangle$

$= \langle 16\cos t\sin^3 t, 32\sin^4 t\rangle$

III. $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

IV. $\int_0^{\pi/2} \langle 16\cos t\sin^3 t, 32\sin^4 t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt$

V. $\int_0^{\pi/2} -32\cos t\sin^4 t + 64\cos t\sin^4 t\, dt$

Well done!

$\int_0^{\pi/2} \cos t\sin^4 t(-32+64)\, dt$

$32\int_0^{\pi/2} \cos t\sin^4 t\, dt$

$32\int_{0}^{1} \cos t\, u^4 \cdot \frac{du}{\cos t} = 32\left[\frac{1}{5}\sin^5 t\right]_0^{\pi/2} = 32\cdot\frac{1}{5} = \frac{32}{5}$

u-sub

Let $u = \sin t$

$du = \cos t\, dt$

$\frac{du}{\cos t} = dt$

$\boxed{\frac{32}{5}}$

6. Show that for any function f(x,y,z) with continuous second partials, curl(grad f) = **0**. Make clear how you use the continuity condition.

$$\text{grad } f = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\text{grad } f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} =$$

$$\langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle$$

Since the second order partials are continuous...

$$f_{xy} = f_{yx}$$

$$f_{xz} = f_{zx}$$

Nice

$$f_{yz} = f_{zy}$$

$$\langle 0, 0, 0 \rangle = \vec{0}$$

7. Biff says "Okay, so I got it figured out. Calculus isn't really math, dude. Math is when, like there's numbers and you work out an answer. On our test, though, the last question was about, like, if you know a vector field is conservative, then what can you say about the line integrals on two legs of a right triangle and the line integral on the hypotenuse. That's obviously, like, a philosophy question or something, not a math question, 'cause it's totally asking for an opinion!"

Explain (clearly enough for Biff to understand) what sort of non-philosophical answer might be given to such a question.

In a conservative vector field, the line integral on any path you take that ends up where it started is 0. A right triangle is one such path: if you trace the outline of a right triangle you end up where you started, regardless of where you started. Naturally

$$\int_{L_1} \vec{F} d\vec{r} + \int_{L_2} \vec{F} d\vec{r} + \int_{Hypoteneuse} \vec{F} d\vec{r} = 0.$$

So, to avoid philosophy, you could say $\int_H \vec{F} d\vec{r} = -\left[\int_{L_1} \vec{F} d\vec{r} + \int_{L_2} \vec{F} d\vec{r}\right]$, or that the line integral of the hypotoneuse is the negative of the sum of the line integrals of the two legs.

Excellent!

8. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a sphere centered at the origin with radius 3 and outward orientation and $\mathbf{F}(x, y, z) = \langle xz, -z, y \rangle$.

It's a closed surface, so let's use the Divergence Theorem!

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(y)$$

$$= z + 0 + 0$$

$$= z$$

So $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E z \, dV$

I recognize this as the numerator for $\bar{z}$, the z-coordinate of the center of mass of this sphere, which must be 0!

$$= 0$$

9. Let $\mathbf{F}(x, y, z) = \langle xz, -z, y\rangle$. Set up an integral for $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the portion with $x \geq 0$ of a sphere centered at the origin with radius 3 and outward orientation.

$x(u,v) = 3\sin u \cos v$

$y(u,v) = 3\sin u \sin v$

$z(u,v) = 3\cos u$

$\vec{r}(u,v) = \langle 3\sin u\cos v, 3\sin u \sin v, 3\cos u\rangle$

$\vec{F}(\vec{r}(u,v)) = \langle 9\sin u\cos u\cos v, -3\cos u, 3\sin u\sin v\rangle$

$\vec{r}_u = \langle 3\cos u\cos v, 3\cos u\sin v, -3\sin u\rangle$

$\vec{r}_v = \langle -3\sin u\sin v, 3\sin u\cos v, 0\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos u\cos v & 3\cos u\sin v & -3\sin u \\ -3\sin u\sin v & 3\sin u\cos v & 0 \end{vmatrix}$$

$$= \langle 0 + 9\sin^2 u\cos v, -(0 + -9\sin^2 u\sin v), 9\sin u\cos u\cos^2 v + 9\sin u\cos u\sin^2$$

$$= \langle 9\sin^2 u\cos v, 9\sin^2 u\sin v, 9\sin u\cos u\rangle$$

So
$$\iint_S \vec{F}\cdot d\vec{S} = \int_{-\pi/2}^{\pi/2}\int_0^{\pi} \langle 9\sin u\cos u\cos v, -3\cos u, 3\sin u\sin v\rangle \cdot \langle 9\sin^2 u\cos v, 9\sin^2 u\sin v, 9\sin u\cos u\rangle \, du\, dv$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi} \left(81\sin^3 u\cos u\cos^2 v - 27\sin^2 u\cos u\sin v + 27\sin^2 u\cos u\sin v\right) du\, dv$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi} 81\sin^3 u\cos u\cos^2 v \, du\, dv$$