Exam 2 Real Analysis 1 11/10/2008

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of compactness.

A set B is compact iff every open cover of B has a finite sub cover.

Yes

2. State the (local) definition of continuity.

Let $f: D \to \mathbb{R}$ and $a \in D$. We say f is continuous at a iff for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|x-a| < \delta$ and $x \in D \Rightarrow |f(x) - f(a)| < \varepsilon$.

Good

3. Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that fails to be differentiable at exactly one point.

$f(x) = |x|$ not differentiable at $x=0$.

It is everywhere else.

Yep.

4. State the Intermediate Value Theorem.

If the function f is continuous on $[a,b]$ and k is a real number between $f(a)$ and $f(b)$, there exists a real number $c \in (a,b)$ such that $f(c) = k$

Good

5. State and prove the Quotient Rule for Derivatives.

f, g are differentiable at a and $g(a) \neq 0$ then $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{g(a)^2}$

Lemma: The inverse rule: I propose $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g(a)^2}$

Proof: Well, $\lim_{x\to a} \frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a}$

$= \lim_{x\to a} \frac{g(a)-g(x)}{g(x)g(a)} \cdot \frac{1}{x-a}$ note that $g(x)$ is non zero in some neighborhood of a and $g(a)$ is not zero.

$= \lim_{x\to a} \frac{g(a)-g(x)}{x-a} \cdot \frac{1}{g(x)g(a)} = -g'(a) \cdot \lim_{x\to a} \frac{1}{g(x)g(a)} = \frac{-g'(a)}{g(a)^2}$

note the derivative exists and thus g is continuous at a.

Well, Now for the proof:

Well by the product rule we get $f'(a)g(a) + f(a)g'(a) = \left(\frac{f}{g}\right)'(a)$

$= f'(a) \cdot \frac{1}{g(a)} + f(a) \frac{-g'(a)}{g(a)^2} = \frac{f'(a)}{g(a)} - \frac{g'(a)f(a)}{g(a)^2} = \frac{f'(a)g(a) - g'(a)f(a)}{g(a)^2}$

Thus the proof is complete.

Good

6. Prove directly from the definition that $f(x) = 5x + 2$ is continuous at $x = 3$.

$f(3) = 17$

for any given $\varepsilon > 0$, we always can find $\delta = \varepsilon/5$ such that $|f(x) - f(a)| < \varepsilon$, provided that $|x - a| < \delta$, $x \in D$

Scratch

$|f(x) - f(a)| < \varepsilon$

$|5x + 2 - 17| < \varepsilon$

$|5(x-3)| < \varepsilon$

$|x-3| < \varepsilon/5$

$|x - 3| < \varepsilon/5$

$5|x-3| < \varepsilon$

$|5x - 15| < \varepsilon$

$|5x + 2 - 17| < \varepsilon$

$\Rightarrow |f(x) - f(3)| < \varepsilon$.

Good

$\Rightarrow$ by our definition $f(x)$ is continuous at $x = 3$.

7. State and prove the Mean Value Theorem.

If a function f is continuous on $[a,b]$ and differentiable on (a,b). there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof

let,

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a} \cdot (x-a)$$

or,

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \cdot 1$$

By Rolle's theorem, we know, there exists a $c \in (a,b)$ such that $g'(c) = 0$

So,

$$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

or,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proved

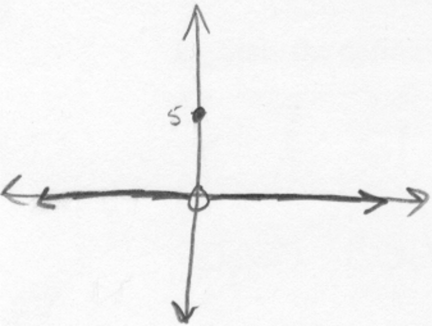
Good

8. Prove that the complement of a closed subset of $\mathbb{R}$ is open.

Let E be the closed subset of $\mathbb{R}$.
S'pose $\mathbb{R} - E$ was not open. Then $\exists\, x \in \mathbb{R} - E$ s.t. no neighborhoods of x are completely within $\mathbb{R} - E$, or every neighborhood of x has an element of E. Then x is an accumulation point of E. But since E is closed, x must be $\in E$. This contradicts that $x \in \mathbb{R} - E$, so $\mathbb{R} - E$ must be open.

Nice!

9. Is the function $f(x)=\begin{cases}0 & \text{if } x\neq 0\\ 5 & \text{if } x=0\end{cases}$ a rational function? Support your answer.



a rational function can be written as a $\dfrac{\text{polynomial}}{\text{polynomial}}$.

polynomials are continuous and differentiable on $\mathbb{R}$

rational functions are differentiable on their domains

f(x) is not differentiable at zero but $0 \in$ of the domain of f(x)

so f(x) is not a rational function

Excellent!

10. Prove or give a counterexample: A function f which is differentiable at $x = a$ must be differentiable on an interval of the form $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$.

Counterexample

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

This function is only differentiable at one point which is $x = 0$.

Excellent

So it is not differentiable on the interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.