Exam 1 Calc 2 9/18/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int x\cos 5x\,dx$.

$u = x \qquad v = \frac{1}{5}\sin 5x$

$du = dx \qquad dv = \cos 5x$

$$\int x\cos 5x\,dx = \frac{x}{5}\sin 5x - \int \frac{1}{5}\sin 5x\,dx =$$

$$\boxed{\frac{x}{5}\sin 5x + \frac{1}{25}\cos 5x + C}$$

Good

check by taking the derivative → $\left(\frac{x}{\cancel{5}}\right)(\cancel{5}\cos 5x) + \frac{1}{5}(\sin 5x) - \left(\frac{1}{\cancel{25}}\right)(\sin 5x)(\cancel{5}) =$

$\checkmark\ \boxed{x\cos 5x} + \cancel{\frac{1}{5}\sin 5x} - \cancel{\frac{1}{5}\sin 5x}$

2. Set up an integral representing the average value of the function $f(x) = 1000 + 48x - 0.024x^2$

2. **Set up** an integral representing the average value of the function $f(x) = 1000 + 48x - 0.034x^2$ on [0, 250].

$a = 0$
$b = 250$

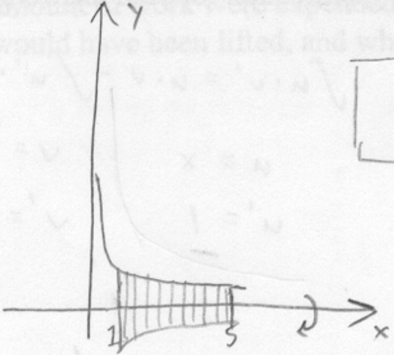
$$\text{avg value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\text{avg value} = \frac{1}{250-0}\int_0^{250} (100 + 48x - .034x^2)\,dx$$

$$\boxed{= \frac{1}{250}\int_0^{250} (100 + 48x - 0.034x^2)\,dx}$$

Great

3. **Set up** an integral for the volume of the solid obtained by revolving around the x axis the region under $y = 1/x$ but above $y = 0$, between $x = 1$ and $x = 5$.



$$\boxed{V = \pi \int_1^5 \left(\frac{1}{x}\right)^2 dx}$$

using disks in x

Great!

4. If $F(t) = \int_{5t}^{\pi} \frac{\sin x}{x} dx$, what is $F'(t)$?

$$F(t) = -\int_{\pi}^{5t} \frac{\sin x}{x} dx$$

So by F.T.C.,

$$F'(t) = -\frac{\sin(5t)}{(5t)} \cdot 5$$

or

$$F'(t) = -\frac{\sin(5t)}{t}$$

.20m

5. If a spring has a natural length of 20cm, and 6J of work is required to stretch it to a length of 40cm, how much work would be required to stretch it from 40cm to 50cm?

.40m .40m .50m [.20, .30]

$$6J = \int_0^{.20} kx\,dx$$

$$6J = \frac{kx^2}{2}\Big|_0^{.20}$$

$$2 \cdot 6J = k(.20)^2$$

$$\frac{12}{.04} = k$$

$$300 = k$$

$$W = \int_{.20}^{.30} 300x\,dx$$

$$W = 150x^2\Big|_{.20}^{.30}$$

$$W = 150[.09 - .04]$$

$$W = \boxed{7.5J}$$

Well done.

6. Evaluate $\int \frac{3x+1}{x^2+1}\,dx$.

$$\int \frac{3x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\frac{1}{2}\int \frac{3}{u}du$$

$$\frac{3}{2}\ln|u| + \tan^{-1}(x) + C$$

Good

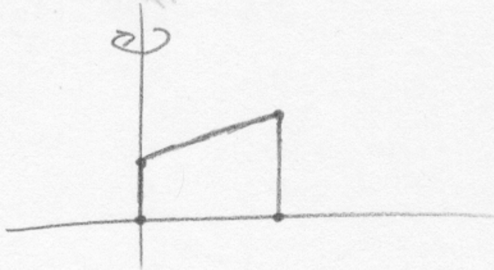
$$\frac{3}{2}\ln|x^2+1| + \tan^{-1}(x) + C$$

$$\frac{3}{2}\ln(x^2+1) + \tan^{-1}(x) + C$$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod! It's, like, sooo unfair! We had to do this thing in our Calc class where, like, instead of being multiple choice it was writing stuff, you know? They said it was for research on how we learn, but I guess they figured out how stupid that was, because they said, like they decided not to grade it for all thousand people in the class because it was too much work, so we got no credit for it after we had to do it anyway. But so anyway the question was, like, if you take the stuff in the trapezee-thingy with corners at (0,0), (5,0), (5,4), and (0,2), and rotate it around the *y* axis thingy, tell at least three ways to figure out the volume you get. So isn't that stupid? In math there's only one way to get the right answer, so how could anybody give three ways?"

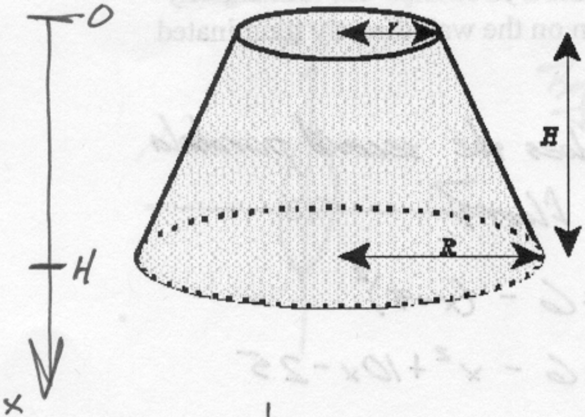
Explain clearly to Bunny some good possible approaches to this problem.

Outstanding!



There are at the very least three good ways of looking at this problem. First, you could consider using the Shell method and do $2\pi \int f(x)\, x \, dx$ where $f(x)$ is the equation of the top-most line. Second you could use a combination of the washer and disk methods to get the volume. This would need to be done with respect to y and be done in two seperate integrals because the washers stop at some point. Thirdly, you could not do any calculus at all and just say its a cylinder $r=5$, $h=4$ minus a cone $h=2$, $r=5$.

8. **Write an integral** for the work required to pump all the water contained in a frustum of a right circular cone with height H, lower base radius R, and top radius r out over the top edge. Assume all measurements are in feet, and that the density of water is 62.5 lb/ft^3.



x	y
0	r
H	R

$$m = \frac{R-r}{H-0} = \frac{R-r}{H}$$

Equation:

$$y - r = \frac{R-r}{H}(x-0)$$

$$y = \frac{R-r}{H}x + r$$

Radius of a Slice $= \left(\frac{R-r}{H}x + r\right)$ ft

Area of a Slice $= \pi\left(\frac{R-r}{H}x + r\right)^2$ ft^2

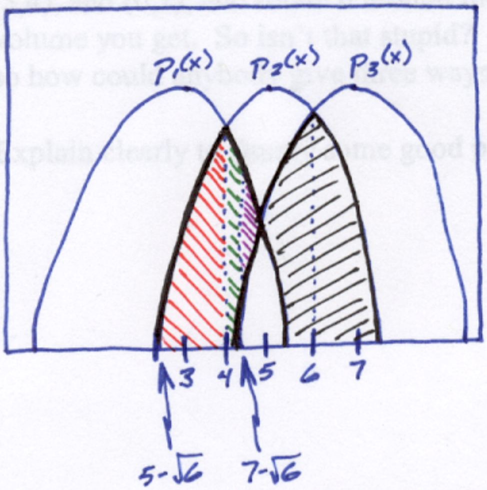
Volume of a Slice $= \pi\left(\frac{R-r}{H}x + r\right)^2 \cdot \Delta x$ ft^3

Weight of a Slice $= \pi\left(\frac{R-r}{H}x + r\right)^2 \Delta x \cdot 62.5$ lbs

Work for a Slice $= 62.5\pi\left(\frac{R-r}{H}x + r\right)^2 \cdot x$ ft·lbs

$$\text{Total Work} = \int_0^H 62.5\pi\left(\frac{R-r}{H}x + r\right)^2 \cdot x\,dx$$

9. Suppose that three ceiling-mounted light fixtures are shining downwards in such a way that they cast light on a nearby wall (where the wall is a rectangle with x values between 0 and 10, and y values between 0 and 8). If the first fixture shines light on the area below $y = 6 - (x - 3)^2$ but above $y = 0$, the second fixture shines light on the area below $y = 6 - (x - 5)^2$ but above $y = 0$, and the third fixture shines light on the area below $y = 6 - (x - 7)^2$ but above $y = 0$, **set up an integral or integrals** for the area of the region on the wall directly illuminated by exactly two of the fixtures.



Let's give these names:

$$p_1(x) = 6 - (x-3)^2$$
$$p_2(x) = 6 - (x-5)^2$$
$$p_3(x) = 6 - (x-7)^2$$

Where do p_1 and p_2 meet?

$$6 - (x-3)^2 = 6 - (x-5)^2$$
$$x^2 - 6x + 9 = x^2 - 10x + 25$$
$$4x = 16$$
$$\therefore x = 4$$

Where does p_2 intersect the x-axis?

$$0 = 6 - (x-5)^2$$
$$0 = x^2 - 10x + 19$$

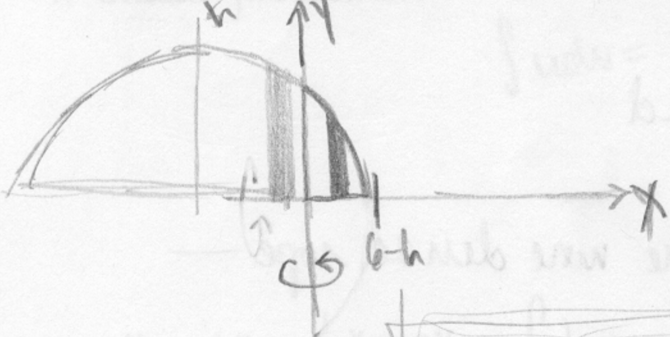
Quadratic Formula:

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(19)}}{2(1)}$$
$$= \frac{10 \pm \sqrt{24}}{2} = 5 \pm \sqrt{6}$$

Now the red/green/purple region is symmetric to the black region, so we'll find the red/green/purple area and double it. We use three integrals since the top and bottom boundary curves change as color-coded in the sketch.

$$\text{Area} = 2\left\{ \int_{5-\sqrt{6}}^{4} p_2(x)\,dx + \int_{4}^{7-\sqrt{6}} p_1(x)\,dx + \int_{7-\sqrt{6}}^{5} [p_1(x) - p_3(x)]\,dx \right\}$$

10. The Nuurf© corporation plans to offer a new line of custom-made toy foam footballs. First the purchaser selects a value of h strictly between 0 and 6 through a cool web-based interface. Then the Nuurf© ball will be formed in a shape like that obtained by taking the region inside $(x + h)^2 + y^2 = 36$ and to the right of the y-axis and rotating it around the y-axis. **Set up an integral** for the volume of a Nuurf© ball as a function of the value of h chosen.



$(x+h)^2+y^2=36$ circle!

$r=6$ $y=\sqrt{36-(x+h)^2}$

$0>h>6$

$$\boxed{2\cdot 2\pi\int_0^{6-h} x\sqrt{36-(x+h)^2}\,dx}$$

Excellent!

the football:

$x+h=\sqrt{36-y^2}$

$x=\sqrt{36-y^2}\ h$

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disics - dy

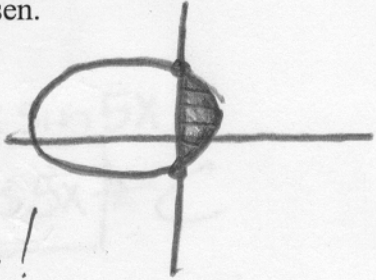
$(x+h)^2+y^2=36$

$(x+h)^2=36-y^2$

$x=\pm\sqrt{36-y^2}-h$

radius $=\pm\sqrt{36-y^2}-h$

area $=\pi\left(\sqrt{36-y^2}-h\right)^2$



Nice!

when $x=0$

$h^2+y^2=36$

$y=\sqrt{36-h^2}$

$$=\boxed{\pi\int_{-\sqrt{36-h^2}}^{+\sqrt{36-h^2}}\left(\sqrt{36-y^2}-h\right)^2\,dy}$$