Exam 2 Calc 2 10/15/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int xe^x \, dx$.

Integration by parts

$\int uv' = uv - \int u'v$

$u = x \quad v = e^x$

$u' = 1 \quad v' = e^x$

$\int xe^x = xe^x - \int e^x$

$\int xe^x = \boxed{xe^x - e^x + C}$

Great!

2. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$\lim_{b\to\infty} -x^{-1}\Big|_1^b$$

$$\lim_{b\to\infty} -\frac{1}{b}+1$$

$$\boxed{1}$$

Good!

3. Write an integral for the length of the curve $f(x) = x^3$ between the points (1,1) and (2,8).

$$\text{Arclength} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$\boxed{C = \int_1^2 \sqrt{1+(3x^2)^2}\,dx}$$

Yes!

4. Evaluate $\int \tan^2 \theta \sec^4 \theta \, d\theta$.

set tos

let $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

$\int \frac{u^2 (\sec^2\theta)^{\cancel{2}}}{\cancel{\sec^2\theta}} du =$

$\int u^2 (u^2+1)\, du =$

$\int u^4 + u^2 \, du =$

$\frac{1}{5}(u^5) + \frac{1}{3}(u^3) + C =$

Good!

$\boxed{\frac{1}{5}(\tan^5\theta) + \frac{1}{3}(\tan^3\theta) + C}$

5. Show that $\int \frac{\sqrt{x+4}}{x}\,dx$ can be transformed into $2\int \frac{u^2}{u^2-4}\,du$.

let $u=\sqrt{x+4} \to u^2 = x+4 \to x = u^2-4$

$du = \frac{1}{2}(x+4)^{-1/2}dx \quad 2du = \frac{dx}{\sqrt{x+4}} \quad dx = 2(\sqrt{x+4})du = 2u\,du$

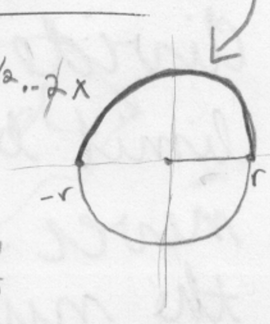
$$\int \frac{\sqrt{x+4}}{x}dx = \int\left(\frac{u}{u^2-4}\right)(2u\,du) = 2\int \frac{u^2}{u^2-4}du$$

Good.

6. Show that the surface area of a sphere with radius r is $4\pi r^2$.

curve: $x^2+y^2=r^2 \to$ top portion & in terms of x $\to y=\sqrt{r^2-x^2}$

Surface Area $= 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$ $\quad y' = \frac{1}{2}(r^2-x^2)^{-1/2}\cdot -2x \quad y' = \frac{-x}{\sqrt{r^2-x^2}}$



change limits to 0 and r instead of -r and r and multiply entire integral by 2

$$2\left[2\pi\int_0^r\left(\sqrt{r^2-x^2}\right)\left(\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\right)dx\right] = 4\pi\int_0^r\left(\sqrt{r^2-x^2}\right)\left(\sqrt{1+\frac{x^2}{r^2-x^2}}\right)dx =$$

$$4\pi\int_0^r\left(\sqrt{r^2-x^2}\right)\left(\sqrt{\frac{r^2-x^2}{r^2-x^2}+\frac{x^2}{r^2-x^2}}\right)dx = 4\pi\int_0^r\left(\sqrt{r^2-x^2}\right)\left(\sqrt{\frac{r^2}{r^2-x^2}}\right)dx = 4\pi\int_0^r\left(\cancel{\sqrt{r^2-x^2}}\right)\left(\sqrt{r^2}\right)\left(\frac{1}{\cancel{\sqrt{r^2-x^2}}}\right)dx$$

$$4\pi r\int_0^r 1\,dx = \boxed{4\pi r^2}$$

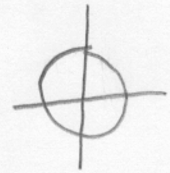
QED!

Very well Done!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, some of this calculus stuff is totally screwed, like they're just trying to make it hard, you know? There was this one problem the guy did in class, and he was, like, making a big deal about it, right? But so he just found the circumference of a circle is $2\pi r$, which I knew from, like, middle school. But so what I really don't get is that he was making a big deal about it being one of the improper integral things, you know? Like that you've gotta do the b approaches something stuff on it? But I don't get that, 'cause I did it without that and got the right answer, and it's not like it had infinity for one of the limits or anything. So why do you have to do that special stuff?"

Explain clearly to Biff why this integral is improper, and also why ignoring that still got him the right answer.

C of a circle = arclength of a circle



circle: $y = \pm\sqrt{r^2 - x^2}$

$y' = \frac{-x}{\sqrt{r^2-x^2}}$

Well Biff, to find that the circumference of a circle is $2\pi r$, you'd need to find the arclengths for various parts of their circle. The equation for arc length is $\sqrt{1+[f'(x)]^2}\,dx$. If you take the derivative of the equation for a circle, you get $f'(x) = \frac{-x}{\sqrt{r^2-x^2}}$. So the integral would end up looking like this $2\int_0^r \sqrt{1+\left(\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,dx$! (The 2 is there because $\frac{x}{\sqrt{r^2-x^2}}$ is the derivative of half of a circle, so it has to be doubled) This seems fine, but if we look closely, we find that there's a hole at the places where $x^2=r^2$ because that would give us a 0 in the denominator, so we have to split up the integral. Ignoring this still gave you the right answer however because it was improper only because of a hole, not because of an asymptote.

Exactly right.

8. Derive Line 37 from the table of integrals.

(37) $$\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + c.$$

$$L.H.S = \int (a^2-u^2)^{3/2}\,du$$

$$= \int (a^2-u^2)\sqrt{a^2-u^2}\,du$$

$$= \int a^2\sqrt{a^2-u^2}\,du - \int u^2\sqrt{a^2-u^2}\,du$$

using line 30

using line 31

Wonderfully Clever!

$$= a^2\int \sqrt{a^2-u^2}\,du - \int u^2\sqrt{a^2-u^2}\,du$$

$$= a^2\left[\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}\right] - \left[\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a}\right]$$

$$= a^2\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^4}{2}\sin^{-1}\frac{u}{a} - \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} - \frac{a^4}{8}\sin^{-1}\frac{u}{a}$$

$$= a^2\frac{u}{2}\sqrt{a^2-u^2} - \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{2}\sin^{-1}\frac{u}{a} - \frac{a^4}{8}\sin^{-1}\frac{u}{a}$$

$$= \sqrt{a^2-u^2}\left[a^2\frac{u}{2} - \frac{u}{8}(2u^2-a^2)\right] + \frac{3}{8}a^4\sin^{-1}\frac{u}{a}$$

$$= \sqrt{a^2-u^2} \times \left[\frac{4a^2u - 2u^3 + a^2u}{8}\right] + \frac{3}{8}a^4\sin^{-1}\frac{u}{a}$$

$$= \sqrt{a^2-u^2} \times \frac{-u}{8}\left[-4a^2 + 2u^2 - a^2\right] + \frac{3}{8}a^4\sin^{-1}\frac{u}{a}$$

$$= \sqrt{a^2-u^2} \times -\frac{u}{8}(2u^2-5a^2) + \frac{3}{8}a^4\sin^{-1}\frac{u}{a}$$

$$\boxed{= \frac{-u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C}$$

∴ L.H.S = R.H.S proved.

9. Evaluate $2\int \frac{u^2}{u^2-4}\,du$. Notice degrees - divide first!

$$2\int \frac{u^2}{u^2-4}\,du = 2\int\left(\frac{u^2-4}{u^2-4}+\frac{4}{u^2-4}\right)du$$

$$= 2\int\left(1+\frac{4}{u^2-4}\right)du$$

I wish:

$$\frac{4}{u^2-4} = \frac{A}{u+2}+\frac{B}{u-2}$$

$$4 = A(u-2)+B(u+2)$$

If $u=2$:

$$4 = 4B \Rightarrow B=1$$

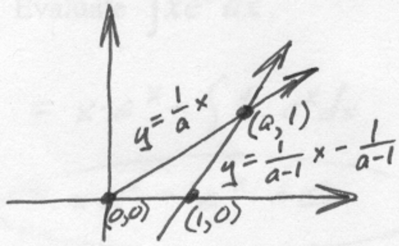
If $u=-2$:

$$4=-4A \Rightarrow A=-1$$

$$= 2\int\left(1+\frac{-1}{u+2}+\frac{1}{u-2}\right)du$$

$$= 2\cdot\left(u-\ln|u+2|+\ln|u-2|\right)+C$$

10. Jon plans to build a triangle. He wants it to have a base of length 1 and height of 1, so he's thinking of it with vertices at (0,0), (1,0), and (a,1). He wants it to be unstable, meaning that the x coordinate of the center of mass must not be located over the base – in everyday terms, he wants it to tip when set on that base. What values of a accomplish this?



$$\bar{x} = \frac{\int_0^a x\cdot\left(\frac{1}{a}x\right)dx - \int_1^a x\cdot\left(\frac{1}{a-1}x - \frac{1}{a-1}\right)dx}{\text{Area of triangle}}$$

$$= \frac{\int_0^a \frac{1}{a}x^2\,dx - \int_1^a\left(\frac{1}{a-1}x^2 - \frac{1}{a-1}x\right)dx}{\frac{1}{2}\cdot(1)(1)}$$

Line through (0,0) and (a,1):

$$m = \frac{1-0}{a-0} = \frac{1}{a}$$

$$y-(0) = \frac{1}{a}(x-(0))$$

$$y = \frac{1}{a}x$$

$$= 2\left[\frac{1}{a}\cdot\frac{x^3}{3}\Big|_0^a - \frac{1}{a-1}\cdot\frac{x^3}{3}\Big|_1^a + \frac{1}{a-1}\cdot\frac{x^2}{2}\Big|_1^a\right]$$

$$= 2\left(\frac{a^3}{3a} - \frac{0}{3a} - \frac{a^3}{3(a-1)} + \frac{1}{3(a-1)} + \frac{a^2}{2(a-1)} - \frac{1^2}{2(a-1)}\right)$$

Line through (1,0) and (a,1):

$$m = \frac{1-0}{a-1} = \frac{1}{a-1}$$

$$y-(0) = \frac{1}{a-1}(x-(1))$$

$$y = \frac{1}{a-1}x - \frac{1}{a-1}$$

$$= 2\left[\frac{a^2}{3} + \frac{1-a^3}{3(a-1)} + \frac{a^2-1}{2(a-1)}\right]$$

$$= 2\left[\frac{a^2}{3} + \frac{(1-a)(1+a+a^2)}{3(a-1)} + \frac{(a-1)(a+1)}{2(a-1)}\right]$$

$$= 2\left(\frac{a^2}{3} + -\frac{1+a+a^2}{3} + \frac{a+1}{2}\right)$$

$$= 2\left(\frac{a^2}{3} - \frac{1}{3} - \frac{a}{3} - \frac{a^2}{3} + \frac{a}{2} + \frac{1}{2}\right)$$

$$= 2\left(\frac{1}{6} + \frac{a}{6}\right)$$

$$= \frac{1}{3} + \frac{a}{3}$$

So if $a > 2$ or $a < -1$, the center of mass won't be over the base and it will be unstable.

$$1-a^3 = (1-a)(1+a+a^2)$$