Exam 4 Calc 2 12/4/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the sum of the series $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\ldots$.

$S=\frac{a}{1-r}$ $a=1$ $r=-\frac{2}{3}$

$S=\frac{1}{1-(-2/3)}=\boxed{\frac{3}{5}}$

Great.

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges using the p-series test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ if $p \leq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

$p > 1$ converges

Excellent!

3. a) Write the MacLaurin polynomial of degree 6 for cos x.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n}}{(2n)!}$$

$$\boxed{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

Good!

b) Determine whether the series $\sum_{n=0}^{\infty} \cos n$ converges or diverges.

$$\lim_{n \to \infty} \cos n \neq 0$$

by the test for divergence it must diverge.

Excellent!

4. Determine whether the series $\sum_{n=2}^{\infty}\frac{1}{n^2-3}$ converges or diverges.

Use the limit comparison test!

compare the series with $\sum 1/n^2$:

$$\lim_{n\to\infty}\left[\frac{1}{n^2-3}\cdot\frac{n^2}{1}\right] = \lim_{n\to\infty}\left[\frac{n^2}{n^2-3}\right] \overset{L'H}{=} \lim_{n\to\infty}\left[\frac{2n}{2n}\right] = \underline{1}$$

Excellent

Since the limit is equal to some finite, positive number ($\underline{1}$), the series behave in the same way. $\sum 1/n^2$ converged as it is a p-series with $p=2$ so also $\sum\frac{1}{n^2-3}$ will $\boxed{\text{converge}}$ by the limit comparison test.

Nicely Done!

5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

use integral test! Excellent!

the corresponding improper integral is:

$$\lim_{c \to \infty} \int_2^c \frac{1}{x \ln x}\,dx$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\lim_{c \to \infty} \int_2^c u^{-1}\,du = \lim_{c \to \infty} \Big[\ln(\ln x)\Big]_2^c = \lim_{c \to \infty} \Big[\ln(\ln c) - \ln(\ln 2)\Big]$$

diverges because as $c \to \infty$, $\ln(c) \to \infty$ and $\ln(\ln c) \to \infty$

since the corresponding integral diverges the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ will also $\boxed{\text{diverge}}$

6. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{1+n!}$ converges or diverges.

use comparison test

I know $0 < 1$ so $n! < 1 + n!$ so $\frac{1}{1+n!} < \frac{1}{n!}$

I also know $\sum \frac{1}{n!}$ converges so since $\frac{1}{1+n!}$ is less than $\frac{1}{n!}$ for all values of n, the series $\sum_{n=0} \frac{1}{1+n!}$ will also $\boxed{\text{converge}}$ by the comparison test.

How do you know that?

Great Job!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, I think they just try really hard to make this series stuff more confusing than it has to be. Like, they tell you twenty different tests when pretty much you only ever use the ratio thing anyway, right? And also they say the ratio thing doesn't tell you anything when it gives a one, right, but I'm pretty sure really those always converge. They just try to confuse us because otherwise everybody would get in on those high-paying math jobs, right?"

Help Biff by addressing his concerns as clearly as possible.

The ratio test isn't perfect for everything. An example of this would be the harmonic series. When you perform the ratio test on it the result will be 1. By your logic that would mean it would converge but we know that not to be the case. The ratio test doesn't work well for everything, p-series often dont work. The test is best for series with factorials or constants raised to the n power. That is where it really gets to shine. As far as knowing what tests to use and when the best advice I can give you is to practice. Experiance is pretty much the only way to know what to use right off the bat and even then you may need to try a few different tests.

Good.

8. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

use ratio test!

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+1}\dfrac{x^{2(n+1)+1}}{2(n+1)+1}\right|}{\left|(-1)^{n}\dfrac{x^{2n+1}}{2n+1}\right|} = \lim_{n\to\infty}\left[\left(\frac{x^{2n+3}}{2n+3}\right)\left(\frac{2n+1}{x^{2n+1}}\right)\right] =$$

$$\lim_{n\to\infty}\left[\frac{x^2(2n+1)}{2n+3}\right] = x^2\left[\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)\right] \overset{L'H}{=}$$

$$x^2\left[\lim_{n\to\infty}\left(\frac{2}{2}\right)\right] = x^2$$

So the ratio test tells that if $x^2 < 1$ then the series will converge. or $|x| < 1$ so the interval of convergence is $-1 < x < 1$ and the radius of convergence is $\boxed{1}$

Good.

9. Determine whether 1 and –1 are included in the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

Let's check $x=1$:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{1}{2n+1}$$

Try A.S.T.: The $(-1)^n$ makes the signs alternate. ✓

$$\lim_{n\to\infty} \frac{1}{2n+1} = 0. \checkmark$$

If $f(x) = \frac{1}{2x+1}$, then $f'(x) = \frac{0\cdot(2x+1) - 1\cdot(2)}{(2x+1)^2}$

$$= \frac{-2}{(2x+1)^2}$$

That's a negative over something non-negative, so it's negative, and since f' is negative we know f is decreasing. ✓

Thus by the A.S.T. we know 1 is included in the I.o.C.

Let's check $x = -1$:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{-1}{2n+1}$$

But that's just the negative of the first part, so it must converge too by the A.S.T.!

Thus both 1 and -1 are in the I.o.C.

10. a) Use a polynomial of degree 4 to approximate $\int_0^{0.1} e^{\left(-x^2\right)}\,dx$.

b) What can you say about how accurate your approximation in part a is, and why?

I know $e^x \approx 1 + x + \frac{x^2}{2}$

so $e^{(-x^2)} \approx 1 - x^2 + \frac{x^4}{2}$

Hence $\int_0^{0.1} e^{(-x^2)}\,dx \approx \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2}\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{0.1}$

$= 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{10}$

$= 0.099667\overline{6}$

If we'd used another term we'd have had

$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$e^{(-x^2)} \approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$

$\int_0^{0.1} e^{(-x^2)}\,dx \approx \int_0^{0.1}\left([\text{as above}] + -\frac{x^6}{6}\right)dx = \left[\text{as above} - \frac{x^7}{42}\right]_0^{0.1}$

$= \text{as above} - \frac{(0.1)^7}{42}$

So since partial sums of an alternating series bound the sum, the real value is between these, and our answer in a is correct to 8 decimal places!