Exam 1　　Calc 3　　9/25/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(x, y) with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Good

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{x^2 + y^2}$ does not exist.

along $x=0$: $\lim_{(0,y)\to(0,0)} \frac{(0)^2-(0)y}{(0)^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{(0)}{y^2} = 0$

along $y=0$: $\lim_{(x,0)\to(0,0)} \frac{x^2+x(0)}{x^2+(0)^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

Excellent!

B/c the limit approaches two diff. values as you move to 0, the limit does not exist.

3. Find the directional derivative of $f(x, y) = 3x^2y - y^4$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

$f_x = 6xy$

$f_y = 3x^2 - 4y^3$

$f_x(2,-1) = -12$

$f_y(2,-1) = 12 - (-4) = 16$

$\nabla f_{(2,-1)} = \langle -12, 16 \rangle$

$\bar{v} = \langle 4, -3 \rangle$

$|\bar{v}| = \sqrt{4^2 + (-3)^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

$\bar{u} = \langle \frac{4}{5}, \frac{-3}{5} \rangle$

Excellent!

$D_{\bar{u}} = \nabla f \cdot \bar{u} = \langle -12, 16 \rangle \cdot \langle \frac{4}{5}, \frac{-3}{5} \rangle = \frac{-48}{5} + \frac{-48}{5} = \boxed{\frac{-96}{5}}$

4. Let $f(x,y) = x^3 + 3xy$. Find the direction in which f is increasing fastest at the point $(-3, 1)$, and the rate of increase in that direction.

$f_x(x,y) = 3x^2+3y$ $\quad f_x(-3,1) = 27+3 = 30$

$f_y(x,y) = 3x$ $\quad f_y(-3,1) = -9$

At $(-3,1)$, f increases fastest in the direction of the vector $\langle 30, -9 \rangle$

Rate of increase $= \sqrt{30^2+(-9)^2} = \sqrt{900+81} = \sqrt{981} = 3\sqrt{109}$

Excellent

5. Jon's daughter Anika has left her Duck Game on again, filling the house with a maddening and incessant "***Quack! Quack! Quack!***" noise. As the battery runs down, the motor also heats up and increases its resistance. Ohm's Law states that $V = IR$. At the moment when $R = 500.\ \Omega$ and $I = 0.080$ A, $dV/dt = -0.020$ V/s and $dR/dt = 0.030\ \Omega$/s. What is dI/dt at this instant? Give your answer correct to at least 6 digits after the decimal point.

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

$V_I = R$

$V_R = I$ $\quad I_V = \frac{1}{R}$

$$\begin{array}{ccc} & V & \\ I & & R \\ | & & | \\ t & & t \end{array}$$

$$-.020 = 500\left(\frac{dI}{dt}\right) + .080(.030)$$

Good

$$-.020 = 500\left(\frac{dI}{dt}\right) + .0024$$

$$-.0224 = 500\left(\frac{dI}{dt}\right)$$

$$\boxed{-.0000448\ \text{A/s}}$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2b_3 - a_3b_2) + \vec{j}(a_3b_1 - a_1b_3) + \vec{k}(a_1b_2 - a_2b_1)$$

$$= \langle \underline{a_2b_3 - a_3b_2},\ \underline{a_3b_1 - a_1b_3},\ \underline{a_1b_2 - a_2b_1} \rangle$$

$$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$$

Cancel cancel

$$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1 = 0$$

cancel

Nice!

When the dot product of two vectors is equal to 0, those vectors are perpendicular.

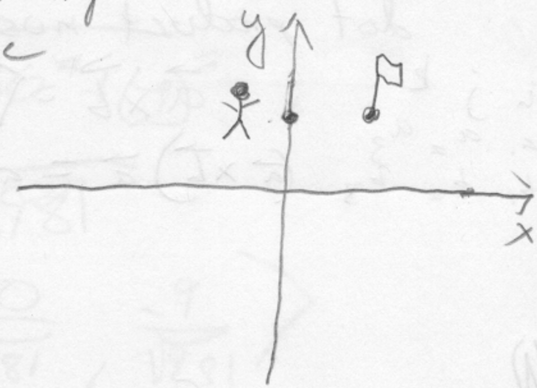
Because $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$, $(\vec{a} \times \vec{b})$ and $\vec{a}$ are perpendicular.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calculus stuff is, like, *sooo* confusing. Like, I understood from before about positive derivatives means increasing and negative derivatives means decreasing, right? But so now I asked our TA if that was the same still, and then it doesn't make sense if f_x is positive but f_y is negative, how can it be both increasing and decreasing at the same time?"

Explain clearly to Bunny how to bring her previous understanding of increasing and decreasing functions to bear on functions of two variables.

Well Bunny, the function in any direction you look is not increasing or decreasing at the same time. f_x refers to, if you are looking just at how the output varies as x varies, but holding y constant. f_y is exactly the opposite. However, if you look at both changes at the same time, you are looking at 2 different components of slope which will look more complicated.

For example pretend you are hiking down a mountain, with the peak to your right. If you only look directly ahead of you, you are traveling down, so a negative f_y. But if you look to your right, the hill goes up, so f_x is positive. Clearly there is no odd space warping with this, just that the slope depends on which way you are looking.



Excellent.

8. Find the extreme values of $f(x, y) = 3x^2 + y^2 - y$ on the disk $x^2 + y^2 \leq 1$.

First find and classify critical points, then use Lagrange's method on the boundary

10

$f_x = 6x$

$f_y = 2y - 1$

$f_{xx} = 6$

$f_{yy} = 2$

$f_{xy} = 0$

$f_x = f_y = 0 \to$

$\begin{cases} 6x = 0 \\ 2y - 1 = 0 \end{cases}$

$6x = 0 \to x = 0$

$2y - 1 = 0 \to$

$2y = 1$

$y = \frac{1}{2}$

critical point: $\underline{(0, \frac{1}{2})}$

$f(0, \frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = \underline{\underline{-\frac{1}{4}}}$

Check: $(0, \frac{1}{2})$ lies in the disk $x^2 + y^2 \leq 1$:

$0^2 + (\frac{1}{2})^2 = \frac{1}{4} < 1$

Excellent!

$\nabla f = \langle 6x, 2y - 1 \rangle \quad g(x,y) = x^2 + y^2$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$g = 1$

$\langle 6x, 2y - 1 \rangle = \lambda \langle 2x, 2y \rangle$

$6x = 2x\lambda$

$2y - 1 = 2y\lambda$

$x^2 + y^2 = 1$

$3x = x\lambda$

$x = 0$ or $\lambda = 3$

$y^2 = 1$ | $2y - 1 = 6y$

$y = \pm 1$ | $-1 = 6y - 2y$

$2 - 1 = 2\lambda$ | $-4y = 1$

or | $y = -\frac{1}{4}$

$-2 - 1 = -2\lambda$

$(0, 1)$ | $x^2 + \frac{1}{16} = 1$

$(0, -1)$ | $x^2 = \frac{15}{16}$

$(\frac{\sqrt{15}}{4}, -\frac{1}{4})$ | $x = \pm \frac{\sqrt{15}}{4}$

$(-\frac{\sqrt{15}}{4}, -\frac{1}{4})$

$f(0, 1) = 1 - 1 = 0$

$f(0, -1) = 1 - (-1) = 2$

$f(\frac{\sqrt{15}}{4}, -\frac{1}{4}) = 3(\frac{15}{16}) + \frac{1}{16} + \frac{1}{4} = \frac{45}{16} + \frac{1}{16} + \frac{4}{16} = \frac{50}{16} =$

$\frac{25}{8} \approx 3.125$

$f(-\frac{\sqrt{15}}{4}, -\frac{1}{4}) = f(\frac{\sqrt{15}}{4}, -\frac{1}{4}) = \frac{25}{8}$

By evenness wrt x (Because f is even with respect to x).

global maxima:
$(\pm \frac{\sqrt{15}}{4}, -\frac{1}{4}, \frac{25}{8})$

global minimum:
$(0, \frac{1}{2}, -\frac{1}{4})$

9. a) Find an equation for the plane tangent to the sphere $x^2 + y^2 + z^2 = 25$ at the point (3, 0, 4).

$f_x = 2x$

$f_y = 2y$

$f_z = 2z$

$f_x(3,0,4) = 6$

$f_y(3,0,4) = 0$

$f_z(3,0,4) = 8$

$6(x-3) + 0(y-0) + 8(z-4) = 0$

$6x - 18 + 8z - 32 = 0$

$6x + 8z = 50$

b) Find an equation for the plane tangent to the sphere $x^2 + y^2 + z^2 = 25$ at the point (3, 4, 0).

$f_x = 2x$

$f_y = 2y$

$f_z = 2z$

$f_x(3,4,0) = 6$

$f_y(3,4,0) = 8$

$f_z(3,4,0) = 0$

$6(x-3) + 8(y-4) + 0(z-0) = 0$

$6x - 18 + 8y - 32 = 0$

$6x + 8y = 50$

Well done!

10. Find the maximum volume of a rectangular box in the first octant with three faces lying in the coordinate planes and one vertex on the plane $x + y + 4z = 4$.

$V = xyz$

$f(x,y,z) = xyz$

Lagrange

$g(x,y,z) = x + y + 4z$

$x + y + 4z = 4$

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla g = \langle 1, 1, 4 \rangle$

$\boxed{\begin{array}{l} x = 4/3 \\ y = 4/3 \\ z = 1/3 \end{array}}$

$\nabla f = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 1, 1, 4 \rangle$

$yz = \lambda$

$xz = \lambda$

$xy = 4\lambda$

$x = \frac{\lambda}{z}$

$x = \frac{4\lambda}{y}$

$\frac{\lambda}{z} = \frac{4\lambda}{y}$

$y\lambda = 4z\lambda$

$y = 4z$

$y = \frac{4\lambda}{x}$ $y = \frac{\lambda}{z}$

$4z\lambda = x\lambda$

$x = 4z$; $z = 1/3$

$x = 4(1/3)$

$x = 4/3$

$x + 4z + 4z = 4$

$12z = 4$

$z = \frac{1}{3}$

Nice Job.

$y = 4z$; $z = 1/3$

$y = 4(1/3)$

$V_{max} = \left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{1}{3}\right) = \cancel{\frac{1}{27}}$

$\frac{4}{3} + \frac{4}{3} + 4\left(\frac{1}{3}\right) = \frac{12}{3} = 4$

$\boxed{V_{max} = \frac{16}{27}}$

10. Find the maximum volume of a rectangular box in the first octant with three faces lying in the coordinate planes and one vertex on the plane $x + y + 4z = 4$.

$$z = \frac{4-x-y}{4}$$

$$V = xy\left(\frac{4-x-y}{4}\right)$$

$$V = \frac{4xy - x^2y - xy^2}{4}$$

$$z = \frac{4-\frac{4}{3}-\frac{4}{3}}{4} = \frac{1}{3}$$

10

$$V_x(x,y) = \frac{4y-2xy-y^2}{4} \qquad V_y(x,y) = \frac{4x-x^2-2xy}{4}$$

$$4y-2xy-y^2=0 \qquad x(4-x-2y)=0$$

if $y=0$

$y=0$ $\quad y(4-2x-y)=0 \qquad x(4-x)=0 \qquad x=0$

$4-2x-y=0 \qquad 4-x=0 \qquad x=4$

$y=4-2x$

If $y=4-2x$

Crits

~~(0,0)~~

~~(0,4)~~

$(4/3, 4/3, 1/3)$

I know none of the ones with 0 are maxes yes!

$$x(4-x-8+4x)=0 \qquad x=0$$

$$4x-x-4=0$$

$$3x-4=0$$

$$x=\frac{4}{3} \qquad x=\frac{4}{3}$$

Nice Job!

So $(4/3, 4/3, 1/3)$ is the max

$$y = 4-\frac{8}{3} = \frac{4}{3}$$

$$V = \frac{16}{27} \text{ units}^3$$