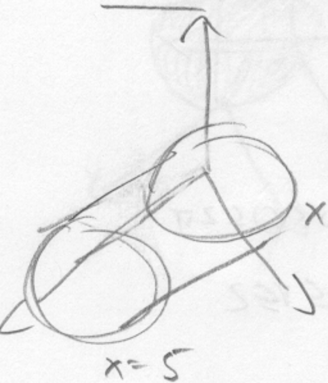
Exam 3 Calc 3 11/24/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Parametrize and give bounds for the portion of the cylinder with radius 2 centered around the x-axis between $x = 0$ and $x = 5$.

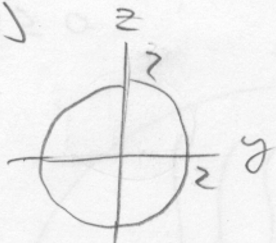


$x(u,v) = v$ $0 \le v \le 5$

$y(u,v) = 2\sin u$ $0 \le u \le 2\pi$

$z(u,v) = 2\cos u$

Okay.

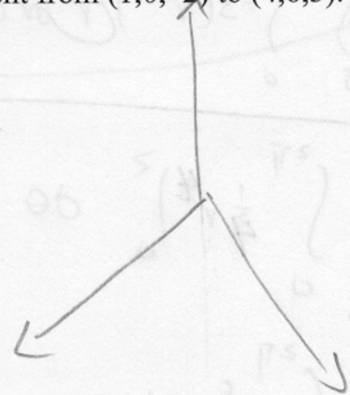


2. Let $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$, and C be the line segment from $(1,0,-2)$ to $(4,6,3)$.
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. line integral, not closed path

So no Greene's, is there

a pot function? Yes!

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$



potential function:

$$\int yz\,dx = xyz$$

$$\int xz\,dy = xyz$$

$$\int xy + 2z\,dz = xyz + z^2$$

$$f(x,y,z) = xyz + z^2$$

$$\text{So } xyz + z^2 \Big|_{(1,0,-2)}^{(4,6,3)}$$

$$\left[(4)(6)(3) + (3)^2\right] - \left[(1)(0)(-2) + (-2)^2\right]$$

$$= [81] - [0 + 4]$$

$$= 81 - 4$$

$$= 77$$

Good

3. Evaluate $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

C is closed and positively oriented, so Green's Theorem can be used.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C y^3\,dx - x^3\,dy = \iint_D \left(-3x^2 - 3y^2\right) dx\,dy$$

circle radius 2, so $0 \le \theta \le 2\pi$ and $0 \le r \le 2$

$$\int_0^{2\pi}\int_0^2 -3r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 -3r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{3}{4}r^4 \Big|_0^2 \,d\theta$$

$$= \int_0^{2\pi} -12\,d\theta$$

$$= -12\theta \Big|_0^{2\pi} = -24\pi$$

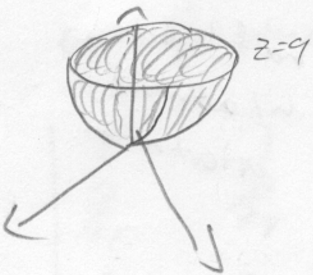
Excellent!

vector

4. Let $\vec{F}(x, y, z) = \langle 2xz, 3z, -z^2 \rangle$, and let S be the paraboloid $z = x^2 + y^2$ below $z = 9$, along with a disc of radius 3 in the plane $z = 9$ centered at (0,0,9), all with outward orientation. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

it's a closed surface

Since we have a disc, use divergence Thm!



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}$$

$$\operatorname{div} \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 2xz, 3z, -z^2 \rangle$$

$$= 2z + 0 - 2z = \underline{0}$$

Excellent!

$$\iiint_E 0 = \underline{0}$$ regardless of the limit of integration

5. Let $\mathbf{G}(x, y, z) = \langle 5x, -z, y\rangle$, and note that curl $\mathbf{G}(x, y, z) = \langle 2,0,0\rangle$. Let S be the portion of $x^2 + y^2 + z^2 = 4$ above $z = 0$, with upward orientation. Evaluate $\iint_S \text{curl}\,\mathbf{G} \cdot d\mathbf{S}$.

Use Stokes Theorem

$\iint_S \text{Curl}\, \vec{G} \cdot ds = \int_C \vec{G} \cdot d\vec{r}$

I $x(t) = 2\cos t$
$y(t) = 2\sin t$
$z(t) = 0$

$r(t) = \langle 2\cos t, 2\sin t, 0\rangle$

II $\vec{F}(\vec{r}(t)) = \langle 10\cos t, 0, 2\sin t\rangle$

III $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

IV $\int_0^{2\pi} \langle 10\cos t, 0, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle \, dt$

V $\int_0^{2\pi} -20\cos t\sin t \, dt$

$u = \cos t$

$du = -\sin t \, dt$

$20\int_0^{2\pi} u \, du = 20\left(\frac{u^2}{2}\right)\Big|_{t=0}^{t=2\pi} = 20\left(\frac{\cos^2 t}{2}\right)\Big|_0^{2\pi}$

Wonderful!

$= 10\cos^2(2\pi) - 10(\cos^2(0))$

$= 10 - 10 = \boxed{0}$

6. Show that for any vector field in $\mathbb{R}^3$ whose component functions have continuous second-order partial derivatives, div curl $\mathbf{F} = 0$. Make it clear why the requirement about continuity is important.

$$\text{div curl}(\vec{F}) = \nabla \cdot (\nabla \times \vec{F})$$

Let $\vec{F} = \langle P, Q, R \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\nabla \cdot (\nabla \times \vec{F}) = \langle \tfrac{d}{dx}, \tfrac{d}{dy}, \tfrac{d}{dz} \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= \frac{d}{dx}(R_y - Q_z) + \frac{d}{dy}(P_z - R_x) + \frac{d}{dz}(Q_x - P_y)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

Excellent!

$$= (R_{yx} - R_{xy}) + (Q_{xz} - Q_{zx}) + (P_{zy} - P_{yz})$$

Because of Clarant's theorem we know $R_{yx} = R_{xy}$ so $R_{yx} - R_{xy} = 0$

and similarly with the others

so $\text{div curl}\,\vec{F} = 0 + 0 + 0 = 0$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. This stuff is so totally confusing. I thought math was where you, like, did problems and stuff, but now there's all this vocabulary stuff, and it's more like French than math or something, you know? So our professor was saying that we should understand why line integrals on <u>closed paths</u> in <u>conservative</u> vector fields are always zero, and I'm not even sure what those words are. Help!"

Explain clearly to Bunny what the underlined terms mean and why the conclusion is valid.

Dear Bunny,

Firstly, a vocabularly lesson. A <u>Closed Path</u> is a vector function which begins and ends in the same coordinates, such that $\vec{r}(a) = \vec{r}(b)$ for $a \leq t \leq b$. A conservative vector field is a vector field which has a potential function f, such that $f_x = P$ and $f_y = Q$. When a vector field is conservative we can use the Fundamental Thrm of Line Integrals which says $f(r(b)) - f(r(a)) = \int_c \vec{F} \cdot d\vec{r}$. Since $b = a$ in a closed path vector function, the FTLI gives $f(r(a)) - f(r(a))$ which $= 0$.

So in conclusion, your professor is right and you should come to Coe.

Wonderful Answer!

8. Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two variables whose gradient vector ∇f is continuous on C. Show that

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

I. $\vec{r}(t) = \langle x(t), y(t) \rangle$

II. $\vec{F}(\vec{r}(t)) = \langle f_x[x(t), y(t)], f_y[x(t), y(t)] \rangle$

III. $\vec{r}'(t) = \langle x'(t), y'(t) \rangle$

IV. $\int_a^b \langle f_x[x(t), y(t)], f_y[x(t), y(t)] \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \right) dt = \int_a^b \frac{\partial f}{\partial t} \, dt$$

$$= f(\vec{r}(t)) \Big|_a^b = f(\vec{r}(b)) - f(\vec{r}(a))$$

Good

9. Let $\mathbf{F}(x, y) = Q(x, y)\,\mathbf{j}$. Let C be the line segment from (c, b) to (d, b). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\vec{F}(x,y) = \langle 0, Q(x,y) \rangle$

I. $x(t) = c + (d-c)t$

$y(t) = b + (b-b)t \Rightarrow b$

$0 \le t \le 1$

$\vec{r}(t) = \langle (c-(d-c)t), b \rangle$

II. $F(\vec{r}(t)) = \langle 0, Q(\vec{r}(t)) \rangle$

III. $\vec{r}\,'(t) = \langle (d-c), 0 \rangle$

Well done!

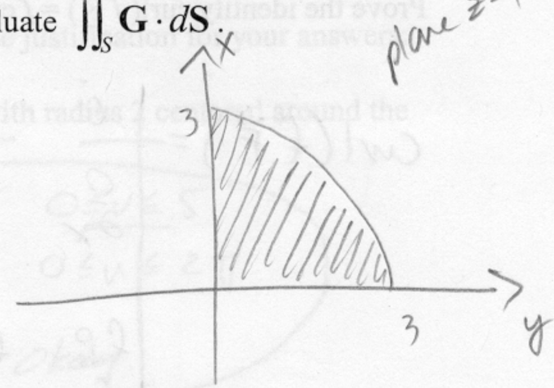
IV $\int_0^1 \langle 0, Q(\vec{r}(t)) \rangle \cdot \langle (d-c), 0 \rangle \, dt$

$\int_0^1 (0 + 0)\,dt$

$= 0$

10. Let $\mathbf{G}(x, y, z) = \langle 2, z, -y\rangle$, and let S be the portion of the plane $z = 1$ in the first quadrant inside the cylinder $x^2 + y^2 = 9$, with upward orientation. Evaluate $\iint_S \mathbf{G}\cdot d\mathbf{S}$.

Surface int the long way b/c not closed and not a curl field



I parameterize: $x(u,v) = v\cos u$
$y(u,v) = v\sin u$
$z(u,v) = 1$

$0 \le v \le 3$
$0 \le u \le \pi/2$

$\vec{r}(u,v) = \langle v\cos u, v\sin u, 1\rangle$

II $\vec{F}(\vec{r}(u,v)) = \langle 2, 1, -v\sin u\rangle$

III $\vec{r}_u = \langle -v\sin u, v\cos u, 0\rangle$
$\vec{r}_v = \langle \cos u, \sin u, 0\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 0 \end{vmatrix}$$

$(0-0)i - (0-0)j + (-v\sin^2 u - v\cos^2 u)$

$= \langle 0, 0, -v(\sin^2 u + \cos^2 u)\rangle = \langle 0, 0, -v\rangle$

WAIT, it's supposed to be upward orientation

so $(-1)\langle 0,0,-v\rangle = \langle 0, 0, v\rangle$

Well done!

IV $$\int_0^{\pi/2}\int_0^3 \langle 2, 1, -v\sin u\rangle \cdot \langle 0, 0, v\rangle \, dv\, du = \int_0^{\pi/2}\int_0^3 -v^2 \sin u \, dv\, du$$

$$= \sin u \int_0^{\pi/2} -\frac{1}{3}v^3\Big|_0^3 dv = \int_0^{\pi/2} -9\sin u \, du = 9\cos u\Big|_0^{\pi/2} = 9[0-1] = -9$$