Exam 1 Calc 3 9/28/2010

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(x, y) with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

Good

2. Show that $\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{x^2+y^2}$ does not exist.

along $x=0$: $\lim_{(0,y)\to(0,0)} \frac{0y-y^2}{0^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

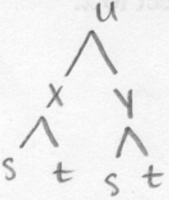
along $y=0$: $\lim_{(x,0)\to(0,0)} \frac{0x-0}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$

Great

$-1 \neq 0$

$\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{x^2+y^2}$ does not exist

3. Write the appropriate version of the chain rule for $\frac{\partial u}{\partial t}$ in the case where $u = f(x, y)$, $x = x(s, t)$, and $y = y(s, t)$. Make clear distinction between derivatives and partial derivatives.



$$\boxed{\frac{\delta u}{\delta t} = \left(\frac{\delta u}{\delta x}\right)\left(\frac{\delta x}{\delta t}\right) + \left(\frac{\delta u}{\delta y}\right)\left(\frac{\delta y}{\delta t}\right)}$$

all are partial derivatives

Great

4. Let $g(x, y) = \cos\sqrt{x^2 + y^2}$. Find the directional derivative of g in the direction of $\langle 2, -1 \rangle$ at the point $(\pi/2, 0)$.

$$g_x = -\sin\left(\sqrt{x^2+y^2}\right) \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$$

$$g_y = -\sin\left(\sqrt{x^2+y^2}\right) \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$$

Excellent!

$$g_x(\pi/2, 0) = -1 \cdot \frac{2}{\pi} \cdot \frac{\pi}{2} = -1 \qquad \nabla g(\pi/2, 0) = \langle -1, 0 \rangle$$

$$g_y(\pi/2, 0) = 0$$

$$D_u = \langle -1, 0 \rangle \cdot \frac{1}{\sqrt{5}} \langle 2, -1 \rangle$$

$$= \frac{1}{\sqrt{5}}(-2 + 0)$$

$$D_u = -\frac{2}{\sqrt{5}}$$

5. Let $f(x,y) = x/y^3$. Find the maximum rate of change of f at the point (2,3) and the direction in which it occurs.

$f(x,y)$ = max. rate of $\quad \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$= \left\langle \frac{1}{y^3}, -3\left(\frac{x}{y^4}\right) \right\rangle$$

$$\nabla f(2,3) = \left\langle \frac{1}{3^3}, -3\left(\frac{2}{(3)^4}\right) \right\rangle$$

Excellent!

$$\boxed{\text{direction of max. rate of change} = \left\langle \frac{1}{27}, \frac{-2}{27} \right\rangle}$$

$|\nabla f(x,y)|$ = max rate of change

$$\sqrt{[f_x(x,y)]^2 + [f_y(x,y)]} = \sqrt{(1/27)^2 + (-2/27)^2} = \sqrt{1/729 + 4/729}$$

max rate of change $= \boxed{\sqrt{5}/27}$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

If the dot product of two nonzero vectors (in this case $\vec{b}$ and $\vec{a} \times \vec{b}$) is zero if and only if those two vectors are perpendicular to each other.

let $\vec{a} = \langle a_1, a_2, a_3 \rangle$
and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2 b_3 \vec{i} - b_2 a_3 \vec{i},\ a_3 b_1 \vec{j} - a_1 b_3 \vec{j},\ a_1 b_2 \vec{k} - b_1 a_2 \vec{k}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle a_2 b_3 - b_2 a_3,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - b_1 a_2 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$

$$= b_1 a_2 b_3 - b_1 b_2 a_3 + b_2 a_3 b_1 - b_2 a_1 b_3 + b_3 a_1 b_2 - b_3 b_1 a_2 = 0$$

Since $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$, $(\vec{a} \times \vec{b})$ is perpendicular to $\vec{b}$

Nice!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. These two other kids in the class seem to know everything, which is totally unfair, so I think we should get graded on a curve, you know? But so they were saying something about how there was this way to do this one problem from the exam, where it was about maxes and mins and stuff, but they said you could do it with gradients. That's total crap, because it's in a different section of the book, you know? So what the heck do gradients have to do with maxes?"

Explain clearly to Biff some significant connection between local optimization and gradients.

Well Biff, we should first understand what gradients are exactly before using them, otherwise unwitting Mountain goats might get hurt. The gradient is a really awesome vector that points to the direction of greatest increase given any function. So,

$f(x,y) = x^2 y \qquad \nabla f = \langle 2xy, x^2 \rangle$ ← This will point to the steepest incline given any x, y.

Now that we know that, where do you suppose it points when we are at the highest point? $\nabla f?$ →

Yes!

That's absolutely right Biff, it is 0 since there is no up to go. So we can use that to help us optimize problems.

8. Find the maximum and minimum values of the function $f(x, y) = 3x^2 + y^2$ on the circle $x^2 + y^2 = 4$.

$f(x,y) = 3x^2 + y^2$

$g(x,y) = x^2 + y^2 = 4$

$\Rightarrow$ lagrange, $\boxed{\begin{array}{c}\nabla f = \lambda \cdot \nabla g \\ g = k\end{array}}$

$\langle 6x, 2y \rangle = \lambda \langle 2x, 2y \rangle$

$6x = 2\lambda x \rightarrow$ (i)

$2y = 2\lambda y \rightarrow$ (ii)

$x^2 + y^2 = 4 \rightarrow$ (iii)

From (i); $6x - 2\lambda x = 0$

or, $2x(3 - \lambda) = 0$

$\therefore x = 0$ or $\lambda = 3$

Putting $x = 0$ in (iii);

$\Rightarrow 0^2 + y^2 = 4$

$y = \pm 2$

from (ii); $2y - 2\lambda y = 0$

or, $2y(1 - \lambda) = 0$

$\therefore y = 0$ or, $\lambda = 1$

Putting $y = 0$ in (iii)

$\Rightarrow x^2 + 0^2 = 4$

$\therefore x = \pm 2$

Well done

$f(0,2) = 4$

$f(0,-2) = 4$

$f(2,0) = 12$

$f(-2,0) = 12$

man = $f(0,2)$

$f(0,-2)$

min = $f(2,0)$

$f(-2,0)$

9. Describe the collection of points on the surface $z = x^2 + y^2$ for which the tangent plane passes through the point (5, 0, 0).

$z_x = 2x$

$z_y = 2y$

So the plane tangent at $(x_0, y_0, x_0^2 + y_0^2)$ is

$$z - (x_0^2 + y_0^2) = 2(x_0)(x - x_0) + 2(y_0)(y - y_0)$$

Now to pass through (5,0,0) would mean

$$(0) - x_0^2 - y_0^2 = 2x_0((5) - x_0) + 2y_0((0) - y_0)$$

$$-x_0^2 - y_0^2 = 10x_0 - 2x_0^2 - 2y_0^2$$

$$0 = 10x_0 - x_0^2 - y_0^2$$

$$x_0^2 - 10x_0 + y_0^2 = 0$$

$$x_0^2 - 10x_0 + 25 + y_0^2 = 25$$

$$(x_0 - 5)^2 + y_0^2 = 25$$

$$\frac{(x_0-5)^2}{25} + \frac{y_0^2}{25} = 1$$

It's a circle centered at (5,0) with radius 5!

10. Let $f(x, y) = x^4 - 5x^2 + y^2 + 2xy$. Find all critical points of this function and classify them as local maxima, local minima, or saddle points.

Take Derivative $f_x = 4x^3 - 10x + 2y$ Set = zero $0 = 2(2x^3 - 5x + y)$

$f_y = 2y + 2x$ $0 = 2(x + y)$

$\Downarrow$

$-x = y$

$\Downarrow$

$0 = 2(2x^3 - 5x + (-x))$

$0 = 2x^3 - 6x$

$0 = 2x(x^2 - 3)$

So $x = 0$ or $x = \pm\sqrt{3}$

$\Rightarrow y = 0$ $y = \mp\sqrt{3}$

$f_{xx} = 12x^2 - 10$

$f_{xy} = 2$

$f_{yy} = 2$

$D(0,0) = (12(0)^2 - 10)(2) - (2)^2$

$= -24 < 0 \Rightarrow$ saddle point

$D(\sqrt{3}, -\sqrt{3}) = (12(\sqrt{3})^2 - 10)(2) - (2)^2$

$= 52 - 4 > 0$

and $f_{xx} = 26 > 0 \Rightarrow$ minimum

$D(-\sqrt{3}, \sqrt{3}) = (12(-\sqrt{3})^2 - 10)(2) - (2)^2$

Ditto