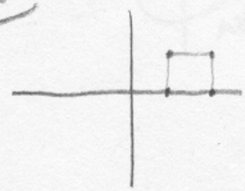
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an iterated integral for the volume under $f(x, y) = x^2 - y^2$ and above the square with vertices (1,0), (2,0), (2,1), and (1,1).

$$\int_0^1 \int_1^2 \int_0^{x^2-y^2} 1 \, dz \, dx \, dy$$

Top View

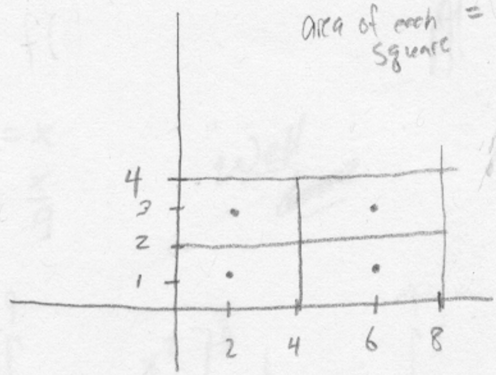


Nice!

2. The table below shows data from a population survey done on brown barbaloots in a region $R = [0,8] \times [0,4]$, given in barbaloots per square furlong. Estimate the total brown barbaloot population in this region using the Midpoint Rule with $m = n = 2$.

x \ y	0	1	2	3	4
0	12	14	15	14	7
2	13	16	21	20	11
4	14	17	23	19	12
6	11	13	21	18	9
8	8	9	10	7	5

area of each square = 8



$f(2,1)(8) + f(6,1)(8) + f(2,3)(8)$

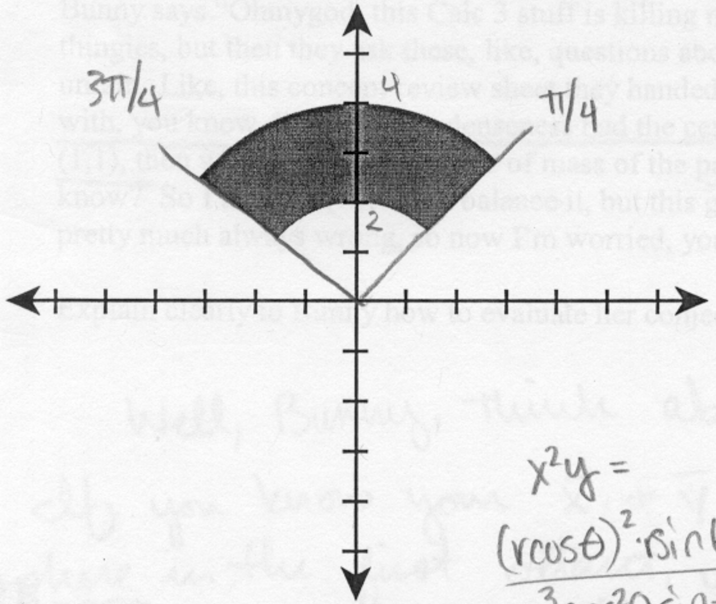
$+ f(6,3)(8)$

$= 16(8) + 13(8) + 20(8) + 18(8)$

$= \boxed{536}$

Great!

3. Set up an iterated integral for the volume below $z = x^2y$, above the region shown below.



Looks like cylindrical...

$$\boxed{\int_{\pi/4}^{\frac{3\pi}{4}}\int_{2}^{4}\int_{0}^{r^3\cos^2\theta\cdot\sin\theta} 1\cdot r\,dz\,dr\,d\theta}$$

Yes!

$$x^2y =$$
$$(r\cos\theta)^2\cdot r\sin\theta$$
$$r^3\cos^2\theta\sin\theta$$

4. Write iterated integrals for the z coordinate of the center of mass of the portion of a sphere (centered at the origin) of radius R lying in the first octant.

Density: $\rho(x,y,z) = k$

$$\bar{z} = \frac{\iiint_S z\,\rho(x,y,z)\,dz\,dx\,dy}{\iiint_S \rho(x,y,z)\,dz\,dx\,dy}$$

$z = \rho\cos\phi$

$$\bar{z} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R K\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \cancel{K}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta} = \boxed{\frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}}$$

Excellent!

5. Set up an iterated integral (with limits, but you don't need to evaluate) for $\iiint_E x^2 e^y \, dV$, where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, and $x = -1$.

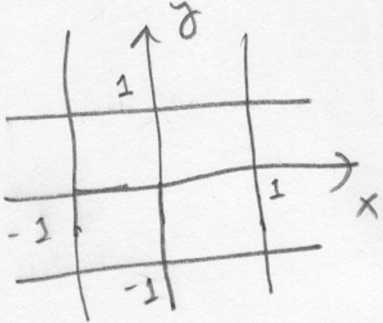
$$\int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1-y^2} x^2 \cdot e^y \, \partial z \, \partial y \, \partial x$$

$z = 1 - y^2$

$1 - y^2 = 0$

$y^2 = 1$

$y = \pm 1$

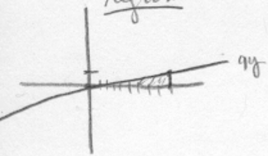


Excellent!

6. Evaluate the iterated integral $\int_0^1 \int_{9y}^9 e^{x^2}\, dx\, dy$.

this would be really nice if it were dy dx, + Fubini said it's okay ☺, so

region



$$\int_0^9 \int_0^{1/9 x} e^{x^2}\, dy\, dx$$

$$e^{x^2} \cdot y \Big|_0^{1/9 x}$$

$$1/9 \int e^{x^2} \cdot x \, dx$$

$u = x^2$

$\frac{1}{2} du = x dx$

$$\frac{1}{9}\left(\frac{1}{2} \int e^u\right) du$$

$$\frac{1}{18} e^u \longrightarrow \frac{1}{18} e^{x^2} \Big|_0^9$$

$$\boxed{\frac{1}{18}\left(e^{81} - 1\right)}$$

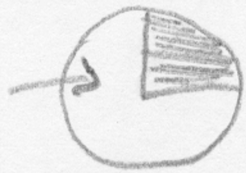
Well done!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is killing me. I can work out all the triple integrand thingies, but then they ask these, like, questions about if you *understand*, you know? It's so unfair. Like, this concept review sheet they handed out had a question about, like, if a circle with, you know, like constant denseness had the center of mass of its first quadrant part at (1,1), then where would the center of mass of the part outside the first quadrant be, you know? So I thought (–1,–1) to balance it, but this guy Biff said that was right too, and he's pretty much always wrong, so now I'm worried, you know?"

Explain clearly to Bunny how to evaluate her conjecture.

Okay, Bunny, chillax for a second. Take a deep breath. Now you got the triple integrals figured out. Good.

Now I'd say a good way to define that other $\frac{3}{4}$ of a circle would be to take the whole thing and subtract the first quadrant...



So with the magical center of mass equation –

$$\bar{x} = \frac{\text{moment(circle)} - \text{moment(1}^{st}\text{ quadrant)}}{\text{area (circle)} - \text{area (1}^{st}\text{ quadrant)}}$$

Nice!

You'd expect the center of mass of that circle to be (0,0), so its moment must be 0, and since the center of that first quadrant is 1, its moment must equal its area!

$$\bar{y} = \bar{x} = \frac{0 - A}{4A - A} = \frac{-A}{3A} = -\frac{1}{3}$$

8. Find the Jacobian for the transformation $x = u/v$, $y = v$.

∂u

∂

$x = u \cdot v^{-1}$ $y = v$

$u(-1 v^{-2})$

$$\text{Jacobian} = \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & 0 \\ (-uv^{-2}) & 1 \end{vmatrix}$$

$$\left(\frac{1}{v}\right)(1) + (u)(0)$$

$$= \frac{1}{v}\, dv\, du$$

Great!

9. Set up integrals for the x coordinate of the center of mass of the first-$\boxed{\text{quadrant}}$ region above the cone $z=\sqrt{x^2+y^2}$ and between the planes $z = 2$ and $z = 5$.

(only top half of cone)

$z^2 = x^2+y^2$

$z^2=x^2+y^2$ When $z=2$: $x^2+y^2=2^2$ circle w/ Radius $=2$

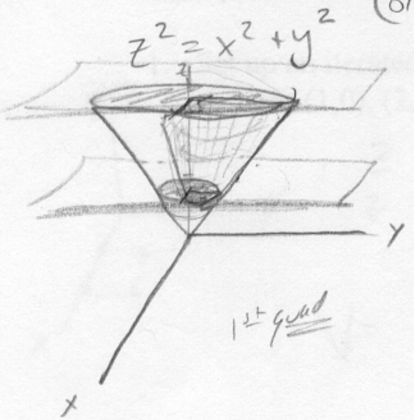
$z=5$: $x^2+y^2=5^2$ circle w/ Radius $=5$

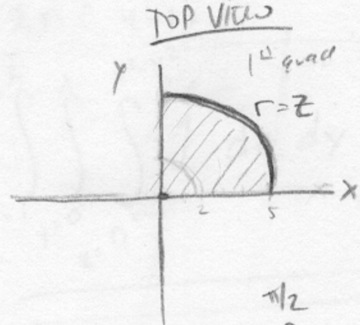
$z=\sqrt{x^2+y^2}$

$z=\sqrt{r^2}$

$z^2=r^2$

$z=r$





In cylindrical:

$r: 0 \to z$

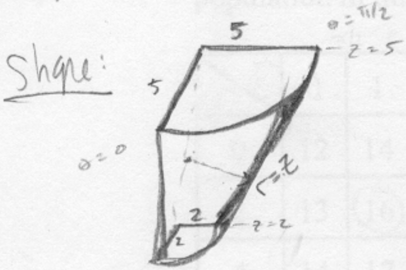
$\theta: 0 \to \frac{\pi}{2}$

$z: 2 \to 5$

$x = r\cos\theta$

$$\overline{X} = \frac{\int_{\theta=0}^{\pi/2}\int_{z=2}^{5}\int_{r=0}^{z} r\cos\theta \, r \, dr\, dz\, d\theta}{\int_{0}^{\pi/2}\int_{2}^{5}\int_{0}^{z} r \, dr\, dz\, d\theta}$$

Shape:



$$\boxed{\overline{X} = \frac{\int_{0}^{\pi/2}\int_{2}^{5}\int_{0}^{z} r^2\cos\theta \, dr\, dz\, d\theta}{\int_{0}^{\pi/2}\int_{2}^{5}\int_{0}^{z} r \, dr\, dz\, d\theta}}$$

Wonderful!

10. The volume of the region in problem 1 turns out to be 2. Are there other points (a, b) in the xy plane such that above a 1 by 1 square with (a, b) as the lower left corner (when viewed from above with the usual orientation) also has a volume under $f(x, y) = x^2 - y^2$ of 2?

$$\int_a^{a+1}\int_b^{b+1} (x^2 - y^2)\,dy\,dx = 2$$

$$\int_b^{b+1} (x^2 - y^2)\,dy = (b+1)x^2 - \frac{(b+1)^3}{3} - bx^2 + \frac{b^3}{3}$$

$$= \cancel{bx^2} + x^2 - \cancel{\frac{b^3}{3}} - b^2 - b - \frac{1}{3} - \cancel{bx^2} + \cancel{\frac{b^3}{3}}$$

$$= x^2 - b^2 - b - \frac{1}{3}$$

All points in a hyperbola centered at $\left(\frac{-1}{2}, \frac{-1}{2}\right)$ with equation $\frac{(a+\frac{1}{2})^2}{2} - \frac{(b+\frac{1}{2})^2}{2} = 1$

$$\int_a^{a+1} \left(x^2 - (b^2 + b + \tfrac{1}{3})\right) dx = 2$$

Exactly so!

$$\frac{(a+1)^3}{3} - \cancel{(b^2+b+\tfrac{1}{3})a} - (b^2+b+\tfrac{1}{3}) - \frac{a^3}{3} + \cancel{(b^2+b+\tfrac{1}{3})a} = 2$$

$$\cancel{\frac{a^3}{3}} + a^2 + a + \cancel{\frac{1}{3}} - b^2 - b - \cancel{\frac{1}{3}} - \cancel{\frac{a^3}{3}} = 2$$

$$a^2 + a - b^2 - b = 2$$

$$(a+\tfrac{1}{2})^2 - (b+\tfrac{1}{2})^2 = 2$$

$$\frac{(a+\frac{1}{2})^2}{2} - \frac{(b+\frac{1}{2})^2}{2} = 1$$

Extra Credit (5 points possible):

Find the volume of the solid bounded between the surfaces