# Exam 3 Calc 3 12/2/2010

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Parametrize and give bounds for the portion of a cylinder centered on the $x$-axis with radius 5 between $x = 0$ and $x = 10$.

$x(u,v) = v$

$y(u,v) = 5\cos u$

$z(u,v) = 5\sin u$

$0 \le u \le 2\pi \qquad 0 \le v \le 10$

Good!

2. Let $\mathbf{F}(x, y, z) = 4x^3yz^2\,\mathbf{i} + (x^4z^2 - 3y^2)\,\mathbf{j} + 2x^4yz\,\mathbf{k}$, and C be the line segment from $(-1,1,2)$ to $(1,3,1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

There is a potential function!

$$\left[x^4yz^2 - y^3\right]_{(-1,1,2)}^{(1,3,1)} \rightarrow$$

Check: $d_x\, x^4yz^2 - y^3 = 4x^3yz^2$ ✓

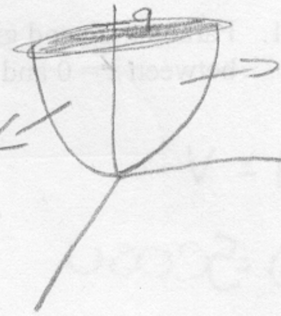
$d_y\, x^4yz^2 - y^3 = x^4z^2 - 3y^2$ ✓

$d_z\, x^4yz^2 - y^3 = 2x^4yz$ ✓

$$\left[(1^4 \cdot 3 \cdot 1^2) - 3^3\right] - \left[((-1)^4 \cdot 1 \cdot 2^2) - 1^3\right] = -24 - 3 = -27$$

Excellent

3. Let $\mathbf{F}(x, y, z) = \langle 2xz, 3z, -z^2 \rangle$, and let S be the paraboloid $z = x^2 + y^2$ below $z = 9$, along with a disc of radius 3 in the plane $z = 9$ centered at (0,0,9), all with outward orientation. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.




$\langle 2xz, 3z, -z^2 \rangle$

closed, so use divergence thm!

$\langle 2xz, 3z, -z^2 \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$= 2z + 0 + -2z = 0$

$\iiint_E 0 \, dA = 0$

Excellent!

limits of integration don't matter!

4. Let $\mathbf{F}(x, y) = \langle -y/2, x/2 \rangle$. Let $C$ be the path parametrized by $\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$ for values of $t$ beginning at 0 and going to $2\pi$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

The long way

I. $\vec{r}(t) = \langle a\cos t, b\sin t \rangle$ for $0 \leq t \leq 2\pi$

II. $F(\vec{r}(t)) = \langle -\frac{b}{2}\sin t, \frac{a}{2}\cos t \rangle$

III. $\vec{r}\,'(t) = \langle -a\sin t, b\cos t \rangle$

IV. $\int_0^{2\pi} \langle -\frac{b}{2}\sin t, \frac{a}{2}\cos t \rangle \cdot \langle -a\sin t, b\cos t \rangle \, dt$

V. $\int_0^{2\pi} \left( \frac{ab}{2}\sin^2 t + \frac{ab}{2}\cos^2 t \right) dt$

$\int_0^{2\pi} \frac{ab}{2} \, dt$

Excellent

$\frac{ab}{2} t \Big|_0^{2\pi} = \boxed{ab\pi}$

5. Let $C$ be the top half of a circle with radius 2 centered at the origin, along with the line segment from (–2,0) to (2,0), all with counterclockwise orientation. Evaluate $\int_C \langle \underset{P}{xy}, \underset{Q}{3y^2 + x^2} \rangle \cdot d\mathbf{r}$.

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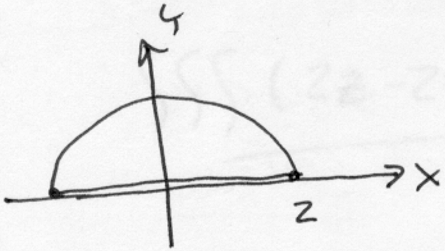



$$\iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint ( 2x - x ) \, dA$$

POLAR

$x = r\cos\theta$ $\quad 0 \le \theta \le \pi$

$y = r\sin\theta$ $\quad 0 \le r \le 2$

$$\int_0^{\pi} \int_0^{2} r\cos\theta \, r \, dr \, d\theta$$

$$\left. \frac{r^3}{3} \right|_0^2 = \frac{8}{3}$$

$$\frac{8}{3} \int_0^{\pi} \cos\theta \, d\theta = \left. \frac{8}{3} \sin\theta \right|_0^{\pi}$$

$$\frac{8}{3} (0 - 0) = \underline{\underline{0}}$$

Well done!

6. Show that for any function $f$ of three variables that has continuous second-order partial derivatives, $\operatorname{curl}(\nabla f) = \mathbf{0}$. Make it clear why the requirement about continuity is important.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\operatorname{curl}(\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

Since their second-order partial derivatives are continuous, Clairaut's Theorem says that,

$$f_{zy} = f_{yz}$$

$$f_{xz} = f_{zx}$$

$$f_{yx} = f_{xy}$$

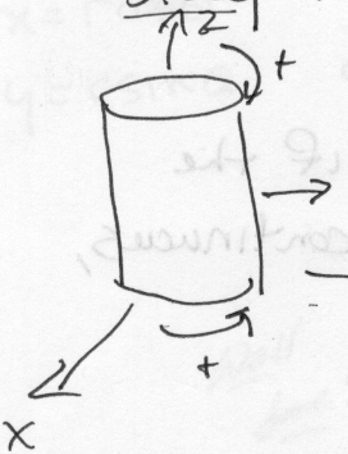
Excellent!

So they will all cancel out giving $\langle 0, 0, 0 \rangle = \vec{0}$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. This stuff is *so* wrong. I thought math was where, like, the answers were numbers, right? But now we're s'posed to be able to do these with this Stokey Theorem, you know? And so the professor said there might be questions for, like, orienting boundaries? Like, he said we should be able to figure out if there was a cylinder centered on the $z$ axis, like from some bottom $z$ to some top $z$, and with the little normal arrow thingies pointing out, then what would its boundary curves be and which way they orient. I liked it better when the answers were things like square root 2, you know?"

Explain clearly to Bunny what the boundary of the surface she describes should be, and why.

the boundary of a surface is where it would bleed if it was cut. In this case it would be the circles at the top and bottom of the cylinder which, given the ~~normal~~ surface normals described ~~who~~ would have positive direction as shown (left toes of the sloth)




*Excellent!*

8. Let $\mathbf{F}(x, y, z) = \langle z - y, x, -x \rangle$, and let $C$ be the circle $x^2 + y^2 = 4$ in the plane $z = 0$, oriented counterclockwise. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$x = u\cos v$
$y = u\sin v$
$z = 0$

This is a surface with C as its boundary

$0 \le u \le 2 \qquad 0 \le v \le 2\pi$

$r_u = \langle \cos v, \sin v, 0 \rangle$

$r_v = \langle -u\sin v, u\cos v, 0 \rangle$

$r_u \times r_v = 0i + 0j + uk$ (because $\cos^2 v + \sin^2 v = 1$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x & -x \end{vmatrix} = \langle 0, 2, 2 \rangle$$

Thus, by Stokes' Theorem,

$$\iint \text{curl} F \cdot r_u \times r_v \, dS = \int_0^{2\pi} \int_0^2 2u \, du \, dv$$

$$\int_0^{2\pi} 4 \, dv = 8\pi$$

Very cool.

9. Let $\mathbf{G}(x, y, z) = \langle 0,2,2\rangle$ and let $S$ be the portion of $x^2 + y^2 + z^2 = 4$ above $z = 0$, with upward orientation. Evaluate $\iint_S \mathbf{G} \cdot d\mathbf{S}$.

$\langle 0,2,2\rangle$ is the curl of the same vector field that I used in problem 8. Thus, use Stokes' Theorem:

$\int_C \langle z-x, x, -x\rangle \cdot dr$ with the same circle as problem 8 as the boundary gets the same question as problem 8.

$8\pi$

Nice!

10. Suppose $\mathbf{F}(x, y) = \langle a, b\rangle$ is a constant vector field (so $a$ and $b$ are constant real numbers). Show that the line integral of $\mathbf{F}$ on a full circle of radius $r$ centered at $(h, k)$ is 0.

Approach #1: $\vec{r}(t) = \langle r\cos t + h, r\sin t + k\rangle$

$\vec{F}(\vec{r}(t)) = \langle a, b\rangle$

$r'(t) = \langle -r\sin t, r\cos t\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle a, b\rangle \cdot \langle -r\sin t, r\cos t\rangle dt$$

$$= \int_0^{2\pi} (-ar\sin t + br\cos t)\,dt$$

$$= \Big[ar\cos t + br\sin t\Big]_0^{2\pi}$$

$$= (ar\cdot 1 + br\cdot 0) - (ar\cdot 1 + br\cdot 0)$$

$$= 0$$

Approach #2: $f(x,y) = ax + by$ is a potential function for $\vec{F}$, so $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$, but since the circle is closed $\vec{r}(b) = \vec{r}(a)$, so we get 0.

Approach #3: It's a closed curve, so use Green's Theorem.

$\frac{\partial b}{\partial x} = 0$ and $\frac{\partial a}{\partial y} = 0$, so $\int_C \vec{F}\cdot dr = \iint_D (0-0)\,dA = \iint_D 0\,dA = 0$ since any integral of 0 gives 0.